# SEMIGROUP IDENTITIES IN THE MONOID OF TWO-BY-TWO TROPICAL MATRICES

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Abstract. We show that the monoid $M_2(\mathbb{T})$ of $2\times 2$ tropical matrices is a regular semigroup satisfying the semigroup identity

$$A^2B^4A^2 \; A^2B^2 \; A^2B^4A^2 = A^2B^4A^2 \; B^2A^2 \; A^2B^4A^2 \; .$$

Studying reduced identities for subsemigroups of $M_2(\mathbb{T})$, and introducing a faithful semigroup representation for the bicyclic monoid by $2\times 2$ tropical matrices, we reprove Adjan's identity for the bicyclic monoid in a much simpler way.

## Introduction

Varieties of semigroups have been intensively studied for many years. It is known that the group of all invertible $2\times 2$ matrices over a field of characteristic 0 contains a copy of the free group and thus does not satisfy any group or semigroup identities. Thus the monoid of all $2\times 2$ over this field generates the variety of all monoids and semigroups and the $2\times 2$ general linear group generates the variety of all groups.

In the last years, tropical mathematics, that is mathematics based upon the tropical semiring, has been intensively studied. In particular, the monoid and semiring of $n\times n$ matrices plays, as one would expect, an important role both algebraically and in applications to combinatorics and geometry. In contrast to the case of matrices over a field, we identify a non-trivial semigroup identity satisfied by the monoid $M_2(\mathbb{T})$ of all $2\times 2$ tropical matrices and for some of its submonoids. We also note that the group of units of this monoid is virtually Abelian and thus both the monoid of all $2\times 2$ tropical matrices and its group of units generate proper varieties of monoids and groups respectively.

Tropical mathematics has been developed mostly over the tropical semiring $\mathbb{T} \;= \mathbb{R}\cup\{-\infty\}$ with the operations of maximum and summation,

$$a \oplus b = \max\{a, b\}, \quad a \odot b = a + b,$$

as addition and multiplication respectively [6, 15, 16]. It is natural for developing the connections between classes of semigroups and their matrix representations by considering matrices over $\mathbb{T}$ as the target for representing semigroups.

One of the fundamental properties of a semigroup is being regular. We prove that $M_2(\mathbb{T})$ is a regular monoid in Von-Neumann's sense, and indicate a naturally occurring generalized inverse for each matrix in $M_2(\mathbb{T})$. We present a few semigroup identities for submonoids of $M_2(\mathbb{T})$ and particularly for $M_2(\mathbb{T})$ itself:

**Theorem 3.6**: *The submonoid $U_2(\mathbb{T}) \subset M_2(\mathbb{T})$ of upper triangular tropical matrices satisfies the semigroup identity*

$$AB^2A \; AB \; AB^2A = AB^2A \; BA \; AB^2A \; ;$$

We note that this is precisely Adjan's identity, the identity of smallest length satisfied by the bicyclic monoid [1]. In fact, we prove that again in contrast with the case of matrices over a field, the bicyclic monoid has a faithful representation in $U_2(\mathbb{T})$. This opens up the possibility of using representation theory over the tropical semiring to study the bicyclic monoid. In particular, we use this faithful tropical linear

*Date*: February 9, 2009.

*Key words and phrases.* Tropical (Max-Plus) Matrix Algebra, Semigroup Identity, Idempotent Semiring, Monoid Representation.

The first author has been supported by the Chateaubriand scientific post-doctorate fellowships, Ministry of Science, French Government, 2007-2008; and was partially supported by a grant from the European Science Foundation (ESF), Automata: from Mathematics to Applications, No. 1609, 2007.

representation for the bicyclic monoid in $U_2(\mathbb{T})$ to reprove Adjan's identity in a much shorter and friendlier way.

Using the above, we prove the main result of the paper.

**Theorem 3.9:** *The monoid $M_2(\mathbb{T})$ admits the semigroup identity*

$$A^2B^4A^2 \ A^2B^2 \ A^2B^4A^2 = A^2B^4A^2 \ B^2A^2 \ A^2B^4A^2 \ .$$

In the past years, most of the theory of matrix semigroups has been developed for matrices built over fields and rings. In this paper we appeal to matrices built over semirings which we believe are the "current" structure to establish representations for classes of semigroups. The bicyclic monoid is one main supporting example.

## 1. Tropical Semirings

We open by reviewing some basic notions of tropical algebra and geometry, including the corresponding categorical framework, and introduce new tropical notions which will be used later in our exposition.

1.1. **Tropical polynomials.** Elements of $\mathbb{T}[\lambda_1, \dots, \lambda_m]$ are called tropical polynomials in $m$ variables over $\mathbb{T}$, and are of the form

$$f = \bigoplus_{\mathbf{i} \in \Omega} \alpha_{\mathbf{i}} \lambda_1^{i_1} \cdots \lambda_m^{i_m} \ \in \ \mathbb{T}[\lambda_1, \dots, \lambda_m] \setminus \{-\infty\}, \tag{1}$$

where $\Omega \subset \mathbb{Z}^{(m)}$ is a finite nonempty set of points $\mathbf{i} = (i_1, \dots, i_m)$ with nonnegative coordinates, $\alpha_{\mathbf{i}} \in \mathbb{R}$ for all $\mathbf{i} \in \Omega$, and $\alpha^i$ means $\alpha \odot \cdots \odot \alpha$ with $\alpha$ repeated $i$ times. Any tropical polynomial determines a piecewise linear convex function $\tilde{f} : \mathbb{R}^{(m)} \to \mathbb{R}$, defined by:

$$\tilde{f}(\mathbf{a}) = \max_{\mathbf{i} \in \Omega} \{ \langle \mathbf{a}, \mathbf{i} \rangle + \alpha_{\mathbf{i}} \}, \quad \mathbf{a} \in \mathbb{R}^{(m)} \ , \tag{2}$$

where $\langle \cdot, \cdot \rangle$ stands for the standard scalar product. The map $f \mapsto \tilde{f}$ is not injective and one can reduce the polynomial semiring so as to have only those elements needed to describe functions.

Given a point $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{T}^{(m)}$, there is a tropical semiring homomorphism

$$\varphi_{\mathbf{a}} : \mathbb{T}[\lambda_1, \dots, \lambda_m] \ \longrightarrow \ \mathbb{T} \tag{3}$$

given by sending

$$\varphi_{\mathbf{a}} : \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} {\lambda_1}^{i_1} \cdots {\lambda_m}^{i_m} \ \longmapsto \ \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} {a_1}^{i_1} \cdots {a_m}^{i_m}$$

which we call the **substitution homomorphism** (relative to $\mathbf{a}$). We write $f(\mathbf{a})$ for the image of $f$ under $\varphi_{\mathbf{a}}$ and identify each $\alpha_{\mathbf{i}} \in \mathbb{T}$ with the monomial $\alpha_{\mathbf{i}} \lambda_1^0 \cdots \lambda_m^0$ to have the embedding $\mathbb{T} \hookrightarrow \mathbb{T}[\lambda_1, \dots, \lambda_m]$.

**Definition 1.1.** *Suppose $f = \bigoplus \alpha_i \lambda_1^{i_1} \cdots \lambda_m^{i_m}$, $h = \alpha_j \lambda_1^{j_1} \cdots \lambda_m^{j_m}$ is a monomial of $f$, and write $f_h = \bigoplus_{i \neq j} \alpha_i \lambda_1^{i_1} \cdots \lambda_m^{i_m}$. We say that the monomial $h$ for $f$ is* ***inessential*** *if $f(\boldsymbol{a}) = f_h(\boldsymbol{a})$ for each $\boldsymbol{a} \in \mathbb{T}^{(m)}$; otherwise $h$ is said to be* ***essential****. The* ***essential part*** *$f^e$ of a polynomial $f = \bigoplus \alpha_i \lambda_1^{i_1} \cdots \lambda_m^{i_m}$ is the sum of those monomials $\alpha_j \lambda_1^{j_1} \cdots \lambda_m^{j_m}$ that are essential. When $f = f^e$, $f$ is said to be an* ***essential polynomial****.*

(Note that, any monomial $\neq -\infty$ by itself, considered as a polynomial, is essential.)

A monomial $h$ is essential in a polynomial $f$ if $h(\mathbf{a}) > f_h(\mathbf{a})$ for some $\mathbf{a}$ and thus for all $\mathbf{a}'$ in an open set $W_{\mathbf{a}}$ of the standard topology of $\mathbb{R}^m$containing $\mathbf{a}$. Any monomial $h$ of $f^e$ is essential in $f^e$. Indeed, by definition, $f_h(\mathbf{a}) \oplus h(\mathbf{a}) > f_h(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{T}^{(m)}$, implying $h(\mathbf{a}) > f_h(\mathbf{a})$.

Using this definition we say that two polynomials $f$ and $g$ are **essentially equivalent**, written $f \overset{e}{\sim} g$, if $f^e = g^e$. We shall show that the essential part of a polynomial $f$ defines the same function as $f$, that is, $f(\mathbf{a}) = g(\mathbf{a})$ for each $\mathbf{a} \in \mathbb{T}^{(m)}$ and is the unique essential polynomial with this property. Thus $f^e$ is a canonical representative for the congruence class of the morphism that sends each polynomial to the function it defines.

**Remark 1.2.** *For any nonconstant monomials $g_1, g_2, h_1, \dots, h_k$ and $\boldsymbol{a} \in \mathbb{T}^{(m)}$ with*

$$g_2(\boldsymbol{a}) = g_1(\boldsymbol{a}) > h_i(\boldsymbol{a}), \quad 1 \le i \le k,$$

*and any open set $W_{\boldsymbol{a}}$ of $\mathbb{T}^{(m)}$, containing $\boldsymbol{a}$, there exists $\boldsymbol{a}' \in W_{\boldsymbol{a}}$ with*

$$g_2(\mathbf{a}') > g_1(\mathbf{a}') > h_i(\mathbf{a}'), \quad 1 \le i \le k.$$

**Lemma 1.3.** *For any monomials $g_1, \dots, g_\ell$, $h_1, \dots, h_k$ and $\boldsymbol{a} \in \mathbb{T}^{(m)}$ with*

$$g_1(\boldsymbol{a}) = g_2(\boldsymbol{a}) = \cdots = g_\ell(\boldsymbol{a}) > h_i(\boldsymbol{a}), \quad 1 \le i \le k,$$

*there exists $\boldsymbol{a}' \in \mathbb{T}^{(m)}$ and $1 < j \le \ell$ such that*

$$g_j(\mathbf{a}') > g_i(\mathbf{a}') \quad \forall i \ne j; \qquad g_j(\mathbf{a}') > h_i(\mathbf{a}'), \quad 1 \le i \le k.$$

*Proof.* Induction on $\ell$. By Remark 1.2, we have $\mathbf{a}' \in \mathbb{T}^{(m)}$ such that

$$g_2(\mathbf{a}') > g_1(\mathbf{a}') > h_i(\mathbf{a}'), \quad 1 \le i \le k.$$

Take $j$ such that $g_j(\mathbf{a}')$ is maximal, and expand the $h_i$ to include all $g_i$ such that $g_j(\mathbf{a}') > g_i(\mathbf{a}')$. Then we have the same hypothesis as before, but with smaller $\ell$. □

**Proposition 1.4.** *$f^e$ defines the same function as $f$ for any $f \in \mathbb{T}[\lambda_1, \dots, \lambda_m]$.*

*Proof.* Given any $\mathbf{a} \in \mathbb{T}^{(m)}$, there is a monomial $g_1$ such that $f(\mathbf{a}) = g_1(\mathbf{a})$. We need to show that $f^e(\mathbf{a}) = g_1(\mathbf{a})$. Suppose $g_1(\mathbf{a}) = g_2(\mathbf{a}) = \cdots = g_\ell(\mathbf{a}) > h(\mathbf{a})$ for some other monomial(s) $g_2, \dots, g_\ell$ of $f$ which are inessential in $f$. But then, by the lemma, we may find $\mathbf{a}'$ such that $g_j(\mathbf{a}')$ takes on the single largest value of the monomials of $f$, for some $2 \le j \le \ell$, contrary to $g_j$ being inessential in $f$. □

**Corollary 1.5.** *$f^e$ is well defined for any polynomial $f$.*

*Proof.* Assume $f_1^e \ne f_2^e$ are two different essential parts of $f$. Then by Proposition 1.4 each of them defines the same polynomial function as $f$. Since $f_1^e \ne f_2^e$, there exists $\mathbf{a} \in \mathbb{T}^{(m)}$ such that $f_1^e(\mathbf{a}) = g_1(\mathbf{a}) \ne h_1(\mathbf{a}) = f_2^e(\mathbf{a})$ for monomials $g_1$ and $h_1$ of $f_1^e$ and $f_2^e$, respectively. Suppose $g_1(\mathbf{a}) > h_1(\mathbf{a})$, then by Lemma 1.2, there is an open set $W \subset \mathbb{T}^{(m)}$ containing $\mathbf{a}$ such that $f_1^e(\mathbf{a}') = g_1(\mathbf{a}') > h_1(\mathbf{a}') = f_2^e(\mathbf{a}')$ for each $\mathbf{a}' \in W$ that is $f_1^e$ defines a different function from $f_2^e$ – a contradiction. □

Therefore, the polynomial semiring $\mathbb{T}[\lambda_1, \dots, \lambda_m]$ can be viewed as the collection of essential polynomials, viewed as a semiring where we perform the usual operations in $\mathbb{T}[\lambda_1, \dots, \lambda_m]$ and then take the essential part. Clearly, $\overset{e}{\sim}$ is an equivalence relation and, for short, we call it $e$-equivalence. It is easy to see that $\overset{e}{\sim}$ is a semiring congruence, and thus we have the quotient semiring $\mathbb{T}[\lambda_1, \dots, \lambda_m]/\overset{e}{\sim}$. Together with Equation (2), this gives the semiring homomorphism

$$\phi : \mathbb{T}[\lambda_1, \dots, \lambda_m]/\overset{e}{\sim} \;\longrightarrow Poly(\mathbb{T}^m),$$

where $Poly(\mathbb{T}^m)$ is the semiring of $\mathbb{T}$-valued polynomial functions.

**Lemma 1.6.** *The homomorphism $\phi$ is an isomorphism. That is, every polynomial function has a unique essential representative.*

*Proof.* Recall that a polynomial function $\tilde{f} : \mathbb{R}^{(m)} \to \mathbb{R}$ of the form (2) determines a piecewise linear convex function whose graph $\Gamma_{\tilde{f}} \subset \mathbb{R}^{(m+1)}$ consists of a finite number of facets (i.e. faces of co-dimension 1). Each facet $\mathcal{F}$ of $\Gamma_{\tilde{f}}$ is described uniquely by a linear equation (in the classical sense), that is a tropical monomial. Accordingly, given a function which is determined by an essential polynomial $f^e$ with $\ell$ monomials, its graph will have exactly $\ell$ facets, each of them is the image of a subset $W_i \subset \mathbb{R}^{(m)}, 1 \le i \le l$ on which the evaluation of $f^e$ is attained by a single monomial which takes maximal values over all the other monomials of $f^e$ on $W$.

Suppose that there are two essential polynomials $f_1^e$ and $f_2^e$ corresponding to the same function $\tilde{f}$ whose graph $\Gamma_{\tilde{f}}$ has $\ell$ facets. Then $f_1^e$ and $f_2^e$ have exactly $\ell$ monomials, each monomial corresponds to a facet $\mathcal{F}$ of $\Gamma_{\tilde{f}}$. Take a facet $\mathcal{F}$. Since it has a unique description each pair of monomials of $f_1^e$ and $f_2^e$ corresponding to $\mathcal{F}$ must be identical. This is true for each facet, and therefore $f_1^e = f_2^e$. □

We say that a polynomial $f$ is a **flat polynomial** if all of its coefficients are equal to some fixed value $\alpha \in \mathbb{T}^\times$, where tropically $\mathbb{T}^\times$ stands for $\mathbb{T} \setminus \{-\infty\}$.

**Lemma 1.7.** *Assume $f = f_1 \oplus f_2 \oplus f_3$ is a flat polynomial in $\mathbb{T}[\lambda_1, \dots, \lambda_m]$, where*

$$f_1 = \lambda_1^{j_1+k}\lambda_2^{j_2-k}\lambda_3^{j_3}\cdots\lambda_m^{j_m}, \qquad f_2 = \lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3}\cdots\lambda_m^{j_m}, \qquad f_3 = \lambda_1^{j_1-k}\lambda_2^{j_2+k}\lambda_3^{j_3}\cdots\lambda_m^{j_m},$$

*are monomials and $k \leq \min\{j_1, j_2\}$ is a non-negative integer. Then $f_2$ is inessential for $f$.*

*Proof.* Pick $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{T}^{(m)}$ and assume $a_1 > a_2$, then $f(\mathbf{a}) = f_1(\mathbf{a}) > f_2(\mathbf{a}), f_3(\mathbf{a})$. Conversely, if $a_2 > a_1$ then $f(\mathbf{a}) = f_3(\mathbf{a}) > f_1(\mathbf{a}), f_2(\mathbf{a})$. When $a_1 = a_2$, $f(\mathbf{a}) = f_1(\mathbf{a}) \oplus f_3(\mathbf{a}) = f_1(\mathbf{a}) \oplus f_2(\mathbf{a}) \oplus f_3(\mathbf{a})$. Namely, $f_2$ is inessential for $f$. $\square$

**Remark 1.8.** *A tropical polynomial $f \in \mathbb{T}[\lambda_1, \dots, \lambda_m]$, written as a sum $\bigoplus_{i=1}^k f_i$ of monomials $f_i$, satisfies the Frobenius property:*

$$\left(\bigoplus_{i=1}^k f_i\right)^n = \bigoplus_{i=1}^k (f_i)^n$$

*for any natural number $n$, cf.* [9, Corollary 3.23]. *In particular, $(a \oplus b)^n = a^n \oplus b^n$, for any $n \in \mathbb{N}$.*

1.2. **Tropical matrices.** It is standard that if $R$ is a semiring then we can form the semiring $M_n(R)$ of all $n \times n$ matrices with entries in $R$, where addition and multiplication are induced from $R$. Accordingly, we define the semiring of tropical matrices $M_n(\mathbb{T})$ over $\mathbb{T} = (\mathbb{T}, \oplus, \odot\ )$, whose unit is the matrix

$$I = \begin{pmatrix} 0 & \dots & -\infty \\ \vdots & \ddots & \vdots \\ -\infty & \dots & 0 \end{pmatrix} \tag{4}$$

and whose zero matrix is $Z = (-\infty)I$. We will consider $M_n(\mathbb{T})$ both as a semiring and also as a multiplicative monoid. We denote tropical matrices as $A = (a_{ij})$ and use the notation $a_{ij}$ for the entries of $A$. Clearly, since $\mathbb{T}$ is commutative, $\alpha A = A\alpha$, for any $\alpha \in \mathbb{T}$ and $A \in M_n(\mathbb{T})$.

**Note 1.9.** *In combinatorics, an $n \times n$ tropical matrix $A \in M_n(\mathbb{T})$ is used to represent a weighted digraph $G = (E, V)$ with $n$ vertices $v_1, \dots, v_n$,* [12]*; we call this digraph the associated digraph of $A$ and denote it by $G_A$. The edges $e \in E$ of $G_A$ are determined by pairs $(v_i, v_j)$ of vertices and the weight $w(e)$ of an edge $e = (v_i, v_j)$ of $G_A$ is the value of the entry $a_{ij}$ in $A$. When $i = j$ the edge $e = (v_i, v_i)$ is called a self loop. Taking a power $A^i$ of $A$ is equivalent to computing all the paths of length $i$ of maximal weights on the associated graph $G_A$ of $A$,* [2].

As usual, we define the ***transpose*** of $A = (a_{ij})$ to be $A^{\mathrm{t}} = (a_{ji})$, and have the usual relation noted here.

**Proposition 1.10.** $(AB)^{\mathrm{t}} = B^{\mathrm{t}}A^{\mathrm{t}}$ .

The proof follow easily from the commutativity and the associativity of $\oplus$ and $\odot$ over $\mathbb{T}$.

The ***minor*** $A_{ij}$ is obtained by deleting row $i$ and column $j$ of $A$. We define the ***tropical determinant*** to be

$$|A| = \bigoplus_{\sigma \in S_n} \left(a_{1\sigma(1)} \cdots a_{n\sigma(n)}\right), \tag{5}$$

where $S_n$ is the set of all the permutations on $\{1, \dots, n\}$. In terms of minors, $|A|$ can be written equivalently as $|A| = \bigoplus_j a_{i_o j}|A_{i_o j}|$, for some fixed index $i_o$. Indeed, in the classical sense, since parity of indices are not involved, the tropical determinant is in fact a permanent, which makes this definition purely combinatorial.

**Remark 1.11.** *When $A$ has either a row (or a column) all of whose entries are $-\infty$ then $|A| = -\infty$ and $|A^i| = -\infty$, for each $i \in \mathbb{N}$, since then $A^i$ also has either a row (or a column) all whose entries are $-\infty$.*

A matrix $A \in M_n(\mathbb{T})$ is said to be **tropically singular** whenever the value of $|A|$, cf. Formula (5), is attained by at least two different permutations $\sigma \in S_n$, otherwise $A$ is called **tropically nonsingular**. The ***adjoint*** matrix $\mathrm{Adj}(A)$ of $A = (a_{ij})$ is defined as the matrix $(a'_{ij})^{\mathrm{t}}$ where $a'_{ij} = |A_{ij}|$. When $|A| \neq -\infty$ we use $A^\nabla$ to denote the tropical quotient

$$A^\nabla := \frac{\mathrm{Adj}(A)}{|A|}.$$

Note that the division in this definition is tropical division, that is, substraction in the usual sense. The ***multiplicative trace*** of $A$ is defined by the following formula.

$$\mathrm{Tr}_{\odot}(A) = \bigodot_i a_{ii} \ , \tag{6}$$

and therefore we always have $|A| \geq \mathrm{Tr}_{\odot}(A)$.

**Proposition 1.12.** *$|A^{n!}| \ = \ \mathrm{Tr}_{\odot}(A^{n!})$ for each $A \in M_n(\mathbb{T})$.*

*Proof.* If $|A| = -\infty$ we are done, cf. Remark 1.11. Write $B = (b_{ij})$ for $A^{n!}$ and assume that $|B| > \mathrm{Tr}_{\odot}(B)$. In view of Note 1.9, the graph $G_B$ of $B$ has a multicycle $C$, of length $n$ whose weight is greater than $\sum_i w(v_i, v_i)$. But since $B = A^{n!}$, the weight of each self loop $(v', v')$ in $G_B$ is the maximal weight over all paths of lengths $\ell \leq n$ from $v'$ to itself in $G_A$ – a contradiction. $\square$

**Example 1.13.** *Let us show by direct computation that $|A^{n!}| \ = \ \mathrm{Tr}_{\odot}(A^{n!})$ for the case of $n = 2$.*

$$\begin{aligned}
\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \right| &= \left| \begin{pmatrix} a^2 \oplus bc & b(a \oplus d) \\ c(a \oplus d) & d^2 \oplus bc \end{pmatrix} \right| \\
&= (a^2 \oplus bc)(d^2 \oplus bc) \oplus bc(a \oplus d)^2 \\
&= (a^2 d^2 \oplus bc(a^2 + d^2) \oplus b^2 c^2) \oplus bc(a \oplus d)^2 \\
&= a^2 d^2 \oplus bc(a^2 + d^2) \oplus b^2 c^2 \qquad\qquad = \mathrm{Tr}_{\odot}(A^2) \ .
\end{aligned}$$

*By the Frobenius property, cf. Remark 1.8, $bc(a \oplus d)^2 = bc(a^2 \oplus d^2)$, so this component becomes inessential and it is omitted.*

**Remark 1.14.** *As a result of Proposition 1.12, we can conclude that not all tropical matrices have a square root; for example take a matrix $A \in M_n(\mathbb{T})$ with diagonal entries $= -\infty$ and all of its off-diagonal entries $\neq -\infty$.*

A set $S$ of vectors $V_1, \dots, V_m \in \mathbb{T}^{(n)}$ is said to be ***linearly dependent*** if there are $\alpha_1, \dots, \alpha_m \in \mathbb{T}$, not all of them $-\infty$, such that each coordinate of the tropical sum $U = \bigoplus_t \alpha_t V_t$ is attained by at least two different terms $\alpha_t V_t$, otherwise $S$ is linearly independent. (In particular, if $m > n$ then $S$ is a dependent set [7].) The ***rank***, $\mathrm{rank}(A)$, of a matrix $A \in M_n(\mathbb{T})$ is the number of elements in a maximal independent subset of rows.

**Theorem 1.15** ([7, Theorem 3.6])**.** *An $n \times n$ matrix $A$ has rank $< n$ iff $A$ is tropically singular.*

Using the theorem, one can easily check whether a matrix $A \in M_n(\mathbb{T})$ has rank $n$ or not. For example, consider a matrix $A \in M_2(\mathbb{T})$ and compute its tropical determinant:

$$\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = ad \oplus bc.$$

If $ad = bc$ then $A$ is tropically singular and thus has rank $< 2$, otherwise $A$ is nonsingular and is of rank 2.

1.3. **Matrices of polynomials and polynomials of matrices.** We also look at the seimiring and monoid of all matrices of polynomials. These are matrices whose entries are elements of the polynomial semiring $\mathbb{T}[\lambda_1, \dots, \lambda_m]$. We denote these matrices by $M_n(\mathbb{T}[\lambda_1, \dots, \lambda_m])$ and for each $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{T}^{(m)}$ we have the induced substitution homomorphism:

$$\varphi_{\mathbf{a}}{}^{\mathrm{mtx}} : M_n(\mathbb{T}[\lambda_1, \dots, \lambda_m]) \ \longrightarrow \ M_n(\mathbb{T}) \tag{7}$$

given by sending $A_{\mathrm{ply}} = (f_{ij}) \mapsto (f_{ij}(\mathbf{a}))$, cf. Formula (3). We write $A_{\mathrm{ply}}(\mathbf{a})$ for the image of $A_{\mathrm{ply}}$ under the substitution homomorphism (to $\mathbf{a} \in \mathbb{T}^{(m)}$). (Clearly, $M_n(\mathbb{T}) \hookrightarrow M_n(\mathbb{T}[\lambda_1, \dots, \lambda_m])$ by sending $A = (a_{ij}) \mapsto (f_{ij})$ where $f_{ij} = a_{ij} \lambda_1^0 \cdots \lambda_m^0$ for all $i, j = 1, \dots, n$.)

Inducing by $e$-equivalence on $\mathbb{T}[\lambda_1, \dots, \lambda_m]$, we say that two matrices $A_{\mathrm{ply}} = (f_{ij})$ and $B_{\mathrm{ply}} = (g_{ij})$ in $M_n(\mathbb{T}[\lambda_1, \dots, \lambda_m])$ are ***essentially equivalent***, denoted as

$$A_{\mathrm{ply}} \ \overset{e}{\sim} \ B_{\mathrm{ply}},$$

if $f_{ij} \overset{e}{\sim} g_{ij}$ for each $i$ and $j$. Therefore, $\overset{e}{\sim}$ is an equivalence relation on $M_n(\mathbb{T}[\lambda_1, \dots, \lambda_m])$ for which $A_{\mathrm{ply}}$ and $B_{\mathrm{ply}}$ are in a same class if and only if $A_{\mathrm{ply}}(\mathbf{a}) = B_{\mathrm{ply}}(\mathbf{a})$ for each $\mathbf{a} \in \mathbb{T}^{(n)}$.

On the other hand, one can talk about polynomials, with coefficients in $\mathbb{T}$, whose arguments are tropical matrices; we denote this polynomial semiring by $\mathbb{T}[\Lambda_1, \dots, \Lambda_m]$ and write $F$, $G$, for its elements. Writing $\mathbf{A}$ for $(A_1, \dots, A_m) \in M_n(\mathbb{T})^{(m)}$, in the standard way, we define the substitution homomorphism

$$\varphi_{\mathbf{A}} : \mathbb{T}[\Lambda_1, \dots, \Lambda_m] \longrightarrow M_n(\mathbb{T}),$$

where $\varphi_{\mathbf{A}} : \mathbf{A} \mapsto F(\mathbf{A})$. Viewing $\Lambda_k = ({\lambda_{ij}}^{(k)})$ as a matrix in $n^2$ indeterminates, we also have the semiring homomorphism

$$\mu_{n,m} : \ \mathbb{T}[\Lambda_1, \dots, \Lambda_m] \longrightarrow M_n(\mathbb{T}[\lambda_1, \dots, \lambda_M]), \quad M = mn^2,$$

given by sending $\mu_{n,m} : \Lambda_k \mapsto ({\lambda_{ij}}^{(k)})$. (Note that $\mu_{n,m}$ is not surjective.) Thus the following diagram commutes.

$$\begin{tikzcd}
\mathbb{T}[\Lambda_1, \dots, \Lambda_m] \arrow[r, "\mu_{n,m}"] \arrow[dr, "\varphi_{\mathbf{A}}"'] & M_n(\mathbb{T}[\lambda_1, \dots, \lambda_M]) \arrow[d, "{\varphi_{\mathbf{a}}}^{\mathrm{mtx}}"] \\
 & M_n(\mathbb{T})
\end{tikzcd} \tag{8}$$

**Remark 1.16.** *Suppose $F, G \in \mathbb{T}[\Lambda_1, \dots, \Lambda_m]$ and consider their images $\mu_{n,m} : F \mapsto A_{\text{ply}}$ and $\mu_{n,m} : G \mapsto B_{\text{ply}}$. In view of Diagram (8), when $A_{\text{ply}} \overset{e}{\sim} B_{\text{ply}}$ then $A_{\text{ply}}(\boldsymbol{a}) = B_{\text{ply}}(\boldsymbol{a})$ for any $\boldsymbol{a} \in \mathbb{T}^{(M)}$, and therefore $F(\boldsymbol{A}) = G(\boldsymbol{A})$ for any $\boldsymbol{A} = (A_1, \dots, A_m) \in M_n(\mathbb{T})^{(m)}$.*

Remark 1.16 plays a key role in this paper and provides the algebraic foundation for our semigroup applications, especially in the study of semigroup identities.

**Remark 1.17.** *A non-essential polynomial $f$, that is, a polynomial $f$ that has an inessential monomial, in $\mathbb{T}[\lambda_1, \dots, \lambda_m]$ can be essential as a polynomial of matrices; for example $\lambda$ is inessential monomial of $\lambda^2 \oplus \lambda \oplus 0$, but the corresponding polynomial $\Lambda^2 \oplus \Lambda \oplus I \in \mathbb{T}[\Lambda]$ is essential, that is $F = \Lambda^2 \oplus \Lambda \oplus I \overset{e}{\nsim} \Lambda^2 \oplus I = G$, since for $A = \begin{pmatrix} -\infty & 0 \\ 0 & -\infty \end{pmatrix}$, $F(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ while $G(A) = A$.*

## 2. The Monoid $M_2(\mathbb{T})$

2.1. **Submonoids and subgroups of $M_n(\mathbb{T})$.** We begin by describing some subsemigroups of $M_n(\mathbb{T})$.

(i) A tropical matrix with each row and column containing exactly one entry $\neq -\infty$ is called a permutation matrix. The set of all tropical permutation matrices, which we denote by $\mathcal{W}_n$, forms the **(affine) Weyl group**. It is the group of units of $M_n(\mathbb{T})$, that is the maximal subgroup with identity element $I$. $\mathcal{W}_n$ contains the Abelian subgroup $\mathcal{D}_n$ of tropical diagonal matrices in $M_n(\mathbb{T})$.

(ii) The upper triangular matrices $U_n(\mathbb{T})$ and the lower triangular matrices $L_n(\mathbb{T})$ are non-commutative submonoids of $M_n(\mathbb{T})$ containing $\mathcal{D}_n$ as a submonoid.

(iii) A matrix, $A \in M_n(\mathbb{T})$ is called **presymmetric** matrix if it is symmetric about its anti-diagonal (i.e secondary diagonal) and is said to be **bisymmetric** if it is both symmetric and presymmetric. It is easy to see that the bisymmetric $2 \times 2$ matrices form a commutative submonoid of $M_2(\mathbb{T})$.

(iv) A tropical matrix $A \in M_2(\mathbb{T})$ with all diagonal entries equal 0 and off-diagonal entries $\leq 0$ is an idempotent matrix, i.e. $A^2 = A$. It is easy to see that the product of any two such matrices is equal to their sum and thus commute. Thus the collection of all such matrices is a submonoid of $M_2(\mathbb{T})$ denoted by $\mathcal{N}_2$. This is a commutative monoid and $|A| = \mathrm{Tr}_{\odot}(A) = 0$ for each $A \in \mathcal{N}_2$.

2.2. **Von-Neumann regularity.** Regularity of semigroup and invertibility of their elements have several different notions, in this paper we use Von-Neumann's notion.

**Definition 2.1.** *Let $(\mathcal{S}, \cdot\ )$ be a semigroup, an element $y \in \mathcal{S}$ is called a **generalized inverse** of $x \in \mathcal{S}$ if*

$$x\,y\,x \ = \ x \quad \textit{and} \quad y\,x\,y \ = \ y \ ,$$

*in the case that $x$ has an inverse we say that $x$ is regular in $\mathcal{S}$. A semigroup $\mathcal{S}$ is said to be a **regular semigroup**, in the Von-Neumann sense, if every element $x \in \mathcal{S}$ has at least one generalized inverse $y \in \mathcal{S}$.*

**Lemma 2.2.** *Each matrix $A \in M_2(\mathbb{T})$ with $|A| = -\infty$ has a generalized inverse.*

*Proof.* Since $|A| = -\infty$, then $A$ has either a row or a column whose entries are all $-\infty$, so it has one of the following forms:

$$A_1 = \begin{pmatrix} a & b \\ -\infty & -\infty \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\infty & -\infty \\ a & b \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & -\infty \\ b & -\infty \end{pmatrix}, \quad A_4 = \begin{pmatrix} -\infty & a \\ -\infty & b \end{pmatrix},$$

where $a, b \in \mathbb{T}$. Correspondingly, we specify their inverses to be:

$$B_1 = \begin{pmatrix} -a & -\infty \\ -b & -\infty \end{pmatrix}, \quad B_2 = \begin{pmatrix} -\infty & -a \\ -\infty & -b \end{pmatrix}, \quad B_3 = \begin{pmatrix} -a & -b \\ -\infty & -\infty \end{pmatrix}, \quad B_4 = \begin{pmatrix} -\infty & -\infty \\ -a & -b \end{pmatrix}.$$

Note that when $a$ or $b$ is $-\infty$, then, respectively, the terms $-a$ or $-b$ are replaced by $-\infty$. $\square$

**Theorem 2.3.** *Assume $|A| \neq -\infty$, then $A^\nabla$ is a generalized inverse of $A \in M_2(\mathbb{T})$.*

*Proof.* Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|A| \neq -\infty$, then $A^\nabla = \begin{pmatrix} d & b \\ c & a \end{pmatrix} / |A|$. Computing the product $AA^\nabla$

$$AA^\nabla = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} / |A| = \begin{pmatrix} 0 & \frac{ab}{|A|} \\ \frac{cd}{|A|} & 0 \end{pmatrix}$$

we get

$$AA^\nabla A = \begin{pmatrix} 0 & \frac{ab}{|A|} \\ \frac{cd}{|A|} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a(0 \oplus \frac{bc}{|A|}) & b(0 \oplus \frac{ad}{|A|}) \\ c(0 \oplus \frac{ad}{|A|}) & d(0 \oplus \frac{bc}{|A|}) \end{pmatrix}.$$

The proof is then derived from the relations $bc, ad \leq |A|$. The relation $A^\nabla A A^\nabla = A^\nabla$ is proved in the same way. $\square$

**Corollary 2.4.** *$M_2(\mathbb{T})$ is a regular semigroup.*

*Proof.* Use Lemma 2.2 and Theorem 2.3 according to when $|A| = -\infty$ or not. $\square$

**Remark 2.5.** *The relations $AA^\nabla A = A$ and $A^\nabla A A^\nabla = A^\nabla$ fail for $A \in M_n(\mathbb{T})$. For example take $A \in M_3(\mathbb{T})$ to be*

$$A = \begin{pmatrix} -4 & 4 & -2 \\ 0 & -1 & -3 \\ 1 & -2 & -3 \end{pmatrix} \quad \textit{then}, \quad |A| = 2 \quad \textit{and} \quad A^\nabla = \begin{pmatrix} -6 & -1 & -1 \\ -4 & -3 & -4 \\ -2 & 3 & 2 \end{pmatrix},$$

*the product is then*

$$AA^\nabla A = \begin{pmatrix} 1 & 4 & -2 \\ 0 & -1 & -3 \\ 1 & -1 & -3 \end{pmatrix} \neq A.$$

*The monoid $M_n(\mathbb{T})$, $n > 2$, is not regular. Indeed, let $\mathbb{B}$ denote the 2 element boolean semiring. Then there is a semiring homomorphism*

$$\psi : \mathbb{T} \longrightarrow \mathbb{B},$$

*given by sending $-\infty \mapsto 0$ and $a \mapsto 1$ for any $a \in \mathbb{R}$, which induces a surjective monoid homomorphism*

$$\psi^{\mathrm{mtx}} : M_n(\mathbb{T}) \longrightarrow M_n(\mathbb{B}).$$

*But it is known that $M_n(\mathbb{B})$ in not regular if $n \geq 3$ (see* [10, Chapter 2]*) and so neither is $M_n(\mathbb{T})$ since regularity is preserved by surjective morphisms.*

**Proposition 2.6.** *When $|A| \neq -\infty$, $AA^\nabla = A^\nabla A$ if and only if $A$ is presymmetric, for any $A \in M_2(\mathbb{T})$.*

*Proof.* Computing the products

$$AA^\nabla = \begin{pmatrix} 0 & \frac{ab}{|A|} \\ \frac{cd}{|A|} & 0 \end{pmatrix} \quad \text{and} \quad A^\nabla A = \begin{pmatrix} 0 & \frac{bd}{|A|} \\ \frac{ac}{|A|} & 0 \end{pmatrix},$$

as far as $|A| \neq -\infty$, one can see that $AA^\nabla = A^\nabla A$ if and only if $a = d$. $\square$

## 3. Semigroup Identities On $M_2(\mathbb{T})$

Our main result in this section is that the monoid $M_2(\mathbb{T})$ admits a non-trivial semigroup identity. We also show that there are semigroup identities for other submonoids of $M_2(\mathbb{T})$ like for triangular tropical matrices.

3.1. **Semigroup identities.** Assuming $(\mathcal{S}, \cdot\ )$ is a semigroup with an identity element 1, we write $x^i$ for the $x \cdot x \cdots x$ repeated $i$ times and identify $x^0$ with 1.

Let $X$ be a countably infinite set of "variables". A semigroup identity is a formal equality $u = v$ where $u$ and $v$ are in the free semigroup $X^+$ generated by $X$. For a monoid identity, we allow $u$ and $v$ to be the empty word as well. A semigroup $S$ satisfies the semigroup identity $u = v$ if for every morphism $f : X^+ \to S$, one has $uf = vf$. Let $I$ be a set of identities. The set of all semigroups satisfying every identity in $I$ is denoted by $V[I]$ and is called the variety of semigroups defined by $I$. It is easy to see that $V[I]$ is closed under subsemigroups, homomorphic images and direct products of its members. The famous Theorem of Birkhoff says that conversely, any class of semigroups closed under these three operations is of the form $V[I]$ for some set of identities $I$.

**Example 3.1.** *The tropical Weyl group $\mathcal{W}_n \subset M_n(\mathbb{T})$ satisfies the identity $A^{n!}B^{n!} = B^{n!}A^{n!}$, since $A^{n!} \in \mathcal{D}_n$ for each $A \in \mathcal{W}_n$ (cf. Proposition 1.12), which is an Abelian group.*

**Remark 3.2.** *The preceding example shows that the group of units of $M_n(\mathbb{T})$ is a virtually Abelian group.*

**Remark 3.3.** *In view of Subsection 1.3, for the case of $\mathcal{S} = M_n(\mathbb{T})$ we can identify any semigroup identity $u = v$ with a pair of monomials $H^u, H^v \in M_n(\mathbb{T})[\Lambda_1, \Lambda_2]$ whose powers are determined by the words $u$ and $v$ and their coefficients are in $M_n(\mathbb{T})$.*

3.2. **The submonoid of triangular matrices.** In this section we prove that

$$AB^2A\ AB\ AB^2A = AB^2A\ BA\ AB^2A\ , \tag{9}$$

is a semigroup identity for the submonoid $U_2(\mathbb{T}) \subset M_2(\mathbb{T})$ of upper triangular tropical matrices.

**Remark 3.4.** *In the case when $A \in M_2(\mathbb{T})$ is of rank 1 it is easy to verify that $A^2 = \alpha A$, for some $\alpha \in \mathbb{T}$; therefore, the semigroup identity (9) is satisfied in $M_2(\mathbb{T})$ whenever $A$ or $B$ is of rank 1. (To see that, just extract the scalar multiplier to obtain the equality.) By the same argument, when $A_{\text{ply}}$ or $B_{\text{ply}}$ is of rank 1, they also satisfy relation (9).*

**Lemma 3.5.** *Suppose $A_{\text{ply}}, B_{\text{ply}} \in U_2(\mathbb{T}[\lambda_1, \dots, \lambda_6])$ are of the form*

$$A_{\text{ply}} = \begin{pmatrix} 0 & \lambda_1 \\ -\infty & \lambda_2 \end{pmatrix}, \qquad B_{\text{ply}} = \begin{pmatrix} 0 & \lambda_3 \\ -\infty & \lambda_4 \end{pmatrix} \tag{10}$$

*then $u(A_{\text{ply}}, B_{\text{ply}}) \overset{e}{\sim} v(A_{\text{ply}}, B_{\text{ply}})$, where $u = AB^2A\ AB\ AB^2A$ and $v = AB^2A\ BA\ AB^2A$.*

*Proof.* Compute the products $F_{\text{ply}} = u(A_{\text{ply}}, B_{\text{ply}})$ and $G_{\text{ply}} = v(A_{\text{ply}}, B_{\text{ply}})$, write $F_{\text{ply}} = (f_{ij})$ and $G_{\text{ply}} = (g_{ij})$, and consider the $f_{ij}$'s and the $g_{ij}$'s as flats polynomials in 4 indeterminates, $\lambda_1, \dots, \lambda_4$. It is easy to verify that

$$f_{11} = g_{11} = 0, \quad f_{21} = g_{21} = -\infty, \quad \text{and} \quad f_{22} = g_{22} = \lambda_2^5\lambda_4^5.$$

Writing $f_{12} = h_{12} \oplus \alpha_{12}$ and $g_{12} = h_{12} \oplus \beta_{12}$, where by direct computation we have,

$$h_{12} = \lambda_1 + \lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4^2 + \lambda_1\lambda_2^3\lambda_4^3 + \lambda_2^4\lambda_3\lambda_4^3 + \lambda_2^4\lambda_3\lambda_4^4 + \lambda_1\lambda_2^4\lambda_4^5\ ,$$

$$\alpha_{12} = \lambda_2^2\lambda_3\lambda_4^2 + \lambda_1\lambda_2^2\lambda_4^3\ , \qquad \text{and} \qquad \beta_{12} = \lambda_1\lambda_2^2\lambda_4^2 + \lambda_2^3\lambda_3\lambda_4^2\ .$$

The diagrams below, together with Lemma 1.7, show that all the terms of $\alpha_{12}$ and $\beta_{12}$ are inessential for $h_{12}$, and thereby $f_{12} \overset{e}{\sim} g_{12}$:

$$\alpha_{12} = \begin{array}{c} \lambda_2\lambda_3\lambda_4 \\ | \\ \lambda_2^2\lambda_3\lambda_4^2 \\ | \\ \lambda_2^4\lambda_3\lambda_4^4 \end{array} \oplus \begin{array}{c} \lambda_1\lambda_2\lambda_4^2 \\ | \\ \lambda_1\lambda_2^2\lambda_4^3 \\ | \\ \lambda_1\lambda_2^4\lambda_4^5 \end{array} \qquad \beta_{12} = \begin{array}{c} \lambda_1 \\ | \\ \lambda_1\lambda_2^2\lambda_4^2 \\ | \\ \lambda_1\lambda_2^3\lambda_4^3 \end{array} \oplus \begin{array}{c} \lambda_2\lambda_3 \\ | \\ \lambda_2^3\lambda_3\lambda_4^2 \\ | \\ \lambda_2^4\lambda_3\lambda_4^3 \end{array}\ .$$

The upper and the lower rows specify the monomials in $h_{12}$ that make respectively each terms of $\alpha_{12}$ and $\beta_{12}$ to be inessential. Taking all together, $f_{ij} \overset{e}{\sim} g_{ij}$ for all $i, j = 1, 2$, and thus $F_{\text{ply}} \overset{e}{\sim} G_{\text{ply}}$. $\square$

**Theorem 3.6.** *The submonoid $U_2(\mathbb{T})$ of upper triangular tropical matrices admits the semigroup identity*

$$AB^2A\ AB\ AB^2A = AB^2A\ BA\ AB^2A\ .$$

*Proof.* Take $A, B \in U_2(\mathbb{T})$, if one of them is of rank 1 we are done by Remark 3.4; otherwise, we can divide $A$ by $a_{11}$ and $B$ by $b_{11}$ to have matrices of the form (10), then in the view of Remark 3.3 the proof is completed by Lemma 3.5. □

**Corollary 3.7.** *The submonoid of lower triangular matrices $L_2(\mathbb{T})$ admits the semigroup identity (9).*

*Proof.* Immediate by Theorem 3.6 and the fact that $L_2(\mathbb{T})$ is conjugate to the monoid $U_2(\mathbb{T})$. □

### 3.3. A semigroup identity on $M_2(\mathbb{T})$.

**Lemma 3.8.** *Suppose $A_{\text{ply}}, B_{\text{ply}} \in M_2(\mathbb{T}[\lambda_1, \dots, \lambda_6])$ are of the form*

$$A_{\text{ply}} = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{pmatrix}, \qquad B_{\text{ply}} = \begin{pmatrix} 0 & \lambda_4 \\ \lambda_5 & \lambda_6 \end{pmatrix} \tag{11}$$

*satisfying the restriction $|A| = \lambda_3$ and $|B| = \lambda_6$, then $u(A_{\text{ply}}, B_{\text{ply}}) \overset{e}{\sim} v(A_{\text{ply}}, B_{\text{ply}})$, where $u = AB^2AABAB^2A$ and $v = AB^2ABAAB^2A$.*

The proof of the lemma is long and technical and is proved in detail as Lemma 4.6 in the Appendix.

**Theorem 3.9.** *The monoid $M_2(\mathbb{T})$ admits the semigroup identity*

$$A^2B^4A^2\ A^2B^2\ A^2B^4A^2 = A^2B^4A^2\ B^2A^2\ A^2B^4A^2\ . \tag{12}$$

*Proof.* Suppose $A, B \in M_2(\mathbb{T})$. If one of these matrices is of rank 1 we are done by Remark 3.4. Otherwise, taking their squares, by Proposition 1.12, $|A^2| = \text{Tr}_{\odot}(A^2)$ and $|B^2| = \text{Tr}_{\odot}(B^2)$, so we can divide their squares to have matrices of the form (11). Consider the matrices over $M_2(\mathbb{T}[\lambda_1, \dots, \lambda_6])$ corresponding to these squares, i.e. matrices of the form (11) whose entries are monomials. In view of Remark 3.3, the proof is then completed by Lemma 3.8. □

## 4. The bicyclic monoid embeds in $M_2(\mathbb{T})$

It is well known that the monoid of matrices $M_n(K)$ over a field $K$ is semisimple [3] and in particular does not have a copy of the bicyclic monoid $\mathcal{B}$ as a subsemigroup. See [13], Chapter 2 for the basic semigroup structure of $M_n(K)$. In contrast to this, we prove in this section that $\mathcal{B}$ has a faithful representation in $M_2(\mathbb{T})$. This explains in part, the identity in Theorem 9. This is Adjan's identity for the bicyclic monoid and is the shortest identity satisfied by $\mathcal{B}$, cf. [1].

Although in this paper we do not study properties of semigroups by their actions on tropical spaces, we open by presenting the tropical analogue of a linear representation.

### 4.1. Tropical Linear Representations.

Considering $\mathbb{T}^{(n)}$ as a space of finite dimension we denote the (tropical) associative semialgebra of all tropical linear operators on $\mathbb{T}^{(n)}$ by $L(\mathbb{T}^{(n)})$. These linear operators can be represented as matrices (in some basis) and this establishes an isomorphism between $L(\mathbb{T}^{(n)})$ and the matrix algebra $M_n(\mathbb{T})$. Therefore, we can identify $L(\mathbb{T}^{(n)})$ with $M_n(\mathbb{T})$. Recall that our ground structure is a semiring and thus, the notions of spaces, operators, and algebra are the corresponding notions [11].

A finite dimensional **tropical linear representation** of a semigroup $\mathcal{M}$, over $\mathbb{T}^{(n)}$, is a semigroup homomorphism

$$R: \ \mathcal{M} \longrightarrow L(\mathbb{T}^n)$$

(The space $\mathbb{T}^{(n)}$ can be replaced by other tropical spaces, but to clarify the exposition we focus on $\mathbb{T}^{(n)}$.) When $R$ is a one-to-one homomorphism, then the representation is called **faithful**. As in classical representation theory one should think of a representation as a tropical linear **action** of $\mathcal{M}$ on $\mathbb{T}^{(n)}$ (since to every $a \in \mathcal{M}$, there is associated a tropical linear operator $R(a)$ which acts on $\mathbb{T}^{(n)}$).

4.2. **A tropical representation of the bicyclic monoid.** The monoid, $\mathcal{B} = \langle a, b \rangle$, generated by two elements $a$ and $b$ satisfying the one relation

$$ab = 1, \tag{13}$$

where 1 is the identity element, is called the **bicyclic monoid**. The elements $x, y \in \mathcal{B}$ of $\mathcal{B}$ are called words (or strings) over $a$ and $b$. It is well known that every element of $\mathcal{B}$ is equal to a unique word of the form $x = b^i a^j, \qquad i, j \in \mathbb{Z}_+$ [3]. As usual, we identify the elements $a^0$ and $b^0$ with the identity element 1 of $\mathcal{B}$.

We start by recalling another representation of the elements of $\mathcal{B}$ which helps us later to formulate a faithful tropical linear representation of $\mathcal{B}$.

Let $\mathcal{S}$ denote $\mathbb{N} \times \mathbb{N}$. We define a binary operation $*$ on $\mathcal{S}$ by the following formula.

$$* : ((i,j),(h,k)) = \begin{cases} (i+h-j, k), & j \le h, \\ (i, j-h+k, ) & j > h, \end{cases} \tag{14}$$

The following proposition is classical [3].

**Proposition 4.1.** *Given a bicyclic monoid $\mathcal{B}$, the map $\phi : \mathcal{B} \to \mathcal{S}$, where $\phi : b^i a^j \mapsto (i,j)$, is a monoid isomorphism.*

*Proof.* Clearly, $\phi$ is a bijective, and

$$\phi((b^i a^j)(b^h a^k)) = \begin{cases} \phi(b^{i+h-j} a^k) \\ \phi(b^i a^{k+j-h}) \end{cases} = \begin{cases} (i+h-j, k), & h \ge j; \\ (i, k+j-h), & h < j; \end{cases} = (\bar{i}, j) * (\bar{h}, k) \ .$$

□

We use this isomorphism to define a tropical linear representation of $\mathcal{B}$. Let $\mathcal{U}_2$ be the subsemigroup of $U_2(\mathbb{T})$, the monoid of $2 \times 2$ upper triangular tropical matrices, generated by the two elements

$$A = \begin{pmatrix} 1^{-1} & 1 \\ -\infty & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ -\infty & 1^{-1} \end{pmatrix}, \tag{15}$$

we write $1^{-1} = \frac{0}{1}$, which is just $-1$ in the usual sense. Having these generators for $\mathcal{U}_2$, $A^j$ and $B^i$, for $i, j \in \mathbb{N}$, can written as

$$A^j = \begin{pmatrix} j^{-1} & j \\ -\infty & j \end{pmatrix}, \qquad B^i = \begin{pmatrix} i & i \\ -\infty & i^{-1} \end{pmatrix},$$

and they satisfy the following relations:

$$E = AB = \begin{pmatrix} 0 & 0 \\ -\infty & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & 2 \\ -\infty & 0 \end{pmatrix}, \qquad B^i A^j = \begin{pmatrix} j^{-1}i & ij \\ -\infty & ji^{-1} \end{pmatrix}, \tag{16}$$

where $i^{-1}j = \frac{j}{i}$ .

**Corollary 4.2.** *$\mathcal{U}_2$ is a bicyclic monoid.*

*Proof.* Immediate from (16) and the fact that $E$ is the identity element of $\mathcal{U}_2$. □

Denoting the tropical semiring, having the addition $\oplus$ and multiplication $\odot$, over $\mathbb{Z} \cup \{-\infty\}$ as $\bar{\mathbb{Z}}$, the operation of the monoid $\mathcal{S}$, cf. Eq. (14), is translated naturally to product of matrices in over $\mathcal{U}_2 \subset U_2(\bar{\mathbb{Z}})$.

**Proposition 4.3.** *The map*

$$\psi \ : \ \mathcal{S} \longrightarrow \mathcal{U}_2, \qquad \psi : (\bar{i}, j) \longmapsto \begin{pmatrix} j^{-1}i & ij \\ -\infty & ji^{-1} \end{pmatrix},$$

*is a monoid isomorphism.*

*Proof.* Assume $\begin{pmatrix} a & b \\ -\infty & c \end{pmatrix} = \begin{pmatrix} j^{-1}i & ij \\ -\infty & ji^{-1} \end{pmatrix}$, then $j = a^{-1}i$, $j = ci$, and $b = ij$. Accordingly, $a = c^{-1}$ and thus $\psi$ is bijective. □

Finally, we define the monoid isomorphism

$$R : \mathcal{B} \longrightarrow \mathcal{U}_2$$

as the composition $R = \psi \circ \phi$ to get:

**Theorem 4.4.** *$R$ is a faithful linear representation of $\mathcal{B}$.*

*Proof.* $R$ is a composition of monoid isomorphisms onto a matrix monoid, and thus is a faithful tropical linear representation. □

**Corollary 4.5.** *The bicyclic monoid $\mathcal{B}$ satisfies the semigroup identity (9), i.e.*

$$xy^2x\ xy\ xy^2x = xy^2x\ yx\ xy^2x$$

*for any $x, y \in \mathcal{B}$.*

*Proof.* Immediate by Corollary 2.4 and Theorem 4.4. □

As mentioned above the semigroup identity in Corollary 4.5 is known as Adjan's identity for the bicyclic monoid [1]. In this paper we have provided an alternative approach for proving this semigroup identity and maybe other semigroup identities. Tropical representation theory thus is useful for studying properties of semigroups where classical representation theory does not have anything to say. A geometric point of view for the identities of a bicyclic monoids is provided in [14].

Note that the morphism in Corollary 4.5 is not a monoid morphism- it does not take the identity element of $\mathcal{B}$ to the identity element of $M_2(\mathbb{T})$. In fact, no such faithful monoid morphism exists. It is not difficult to see that the $\mathcal{D}$-class of 1 in $M_2(\mathbb{T})$ is the Weyl group $\mathcal{W}_n$. Since $\mathcal{B}$ is a bisimple monoid that is not a group, no faithful monoid morphism exists. See [8] for more information on Green's relations in full tropical matrix monoids.

4.3. **Remarks and Open Problems.** We have proved that $M_2(\mathbb{T})$ satisfies a non-trivial semigroup identity and that begs the question about whether $M_n(T)$ satisfies non-trivial identities for all $n > 2$. We conjecture that this is so, but have not been able to prove this as of yet.

For $n = 2$, the connection between the monoid of upper triangular $2 \times 2$ tropical matrices and the bicyclic monoid is deeper than we've indicated in this paper. In fact the monoid of upper triangular matrices of rank 2 form an inverse monoid that is isomorphic to the monoid of partial shifts of the real line, just like $\mathcal{B}$ is isomorphic to the monoid of partial shifts on the natural numbers [8]. It was this connection that lead us to try Adjan's identity on the submonoid of upper triangular matrices in $M_2(\mathbb{T})$ and eventually to the identity for all of $M_2(\mathbb{T})$ that we found in this paper. We would like to clarify the exact relationship between the bicyclic monoid and the monoid of upper triangular $2 \times 2$ matrices further. We ask if they generate the same variety, that is, if they satisfy exactly the same identities.

The structure of upper triangular full rank $n \times n$ tropical matrices is illimunated in [8]. It is a block group, that is a monoid in which each $\mathcal{R}$ and $\mathcal{L}$ class have at most one idempotent, but is not an inverse monoid if $n > 2$. This has made finding an identity difficult computationally for this monoid. Passing to the monoid of all $n \times n$ matrices is also difficult.

Another reason for conjecturing that $M_n(\mathbb{T})$ satisfies a non-trivial identity for all $n$ is that every finite subsemigroup of $M_n(\mathbb{T})$ has polynomial growth [4, 18]. In particular, the free semigroup on 2 generators is not isomorphic to a subsemigroup of $M_n(\mathbb{T})$. While Shneerson [17] has given examples of polynomial growth semigroups that do not satisfy any non-trivial identity (no such example exists for groups by Gromov's Theorem [5], we feel that this is not the case for $M_n(\mathbb{T})$.

## References


[1] S. I. Adjan, *Defining relations and algorithmic problems for groups and semigroups.* Number 85. Proceeding of the Steklov Institute of Mathematics, American Mathematical Society, 1967.

[2] R. A. Brualdi and H. J. Ryser, *Combinatorial matrix theory.* Cambridge University Press, 1991.

[3] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, AMS,Providence, R.I., , Volume 1, 1961, Volume 2, 1967.

[4] S. Gaubert and R.D. Katz, Reachability problems for products of matrices in semirings, *Int. J. of Alg. and Comp.*, Vol. 16 no. 3(2006), 603-627.

[5] *Groups of polynomial growth and expanding maps* Publ. Math IHES 53 (1981), 53-73.

[6] I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*, volume 35. Birkhauser, 2007. Oberwolfach seminars.

[7] Z. Izhakian. The tropical rank of a tropical matrix. Preprint at arXiv:math.AC/0604208, 2005.
[8] Z. Izhakian and S.W. Margolis *Green's Relations on the monoid of all tropical matrices*. To appear.
[9] Z. Izhakian and L. Rowen, Supertropical algebra, preprint at arXiv:0806.1175, 2007.
[10] K. H. Kim. *Boolean Matrix Theory and Applications*, volume 70 of *Monographs and Textbooks in Pure and Applied*. Marcel Dekker, New York, 1982.
[11] G. Lallement. *Semigroups and Combinatorial Applications*. John Wiley & Sons, Inc., New York, NY, USA, 1979.
[12] E. L. Lawler. *Combinatorial Optimization: Networks and Matroids*. Holt, Rinehart, and Winston, 1976.
[13] J.Okninski, *Semigroups of Matrices* World Scientific, Singapore, 1998.
[14] F. J. Pastijn. Polyhedral convex cones and the equational theory of the bicyclic semigroup. *J. Austra. Math. Soc*, 81:63–96, 2006.
[15] J.-E. Pin. Tropical semirings. *Cambridge Univ. Press, Cambridge*, 11:50–69, 1998. Publ. Neton Inst. 11, Cambridge Univ.
[16] J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. *Idempotent mathematics and mathematical physics*, pages 289–317, 2005. Contemp. Math., Amer. Math. Soc., Providence, RI, 377.
[17] L. Shneerson *Identities in finitely generated semigroups of polynomial growth* J. Algebra 154 (1993), no. 1, 67–85.
[18] I. Simon, *Recognizable sets with multiplicities in the tropical semiring*, in MFCS 88, editors, M. Chytil, L. Janiga, V. Koubek, Lecture Notes in Computer Science, Number 324, Springer, 107-120, 1988.

## Appendix A

To clarify the exposition, instead of $\lambda_1, \dots, \lambda_6$, we use the letters $a, b, c, x, y,$ and $z$, to denote the variables of matrices of polynomials $A_{\mathrm{ply}}, B_{\mathrm{ply}} \in M_2(\mathbb{T}[\lambda_1, \dots, \lambda_6])$.

**Lemma 4.6.** *Given two matrices $A_{\mathrm{ply}}, B_{\mathrm{ply}} \in M_2(\mathbb{T}[a, b, c, x, y, z])$ of the form*

$$A_{\mathrm{ply}} = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \qquad \text{and} \qquad B_{\mathrm{ply}} = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix},$$

*assuming $|A_{\mathrm{ply}}| = c$ and $|B_{\mathrm{ply}}| = z$, then*

$$A_{\mathrm{ply}} B_{\mathrm{ply}}^2 A_{\mathrm{ply}}^2 B_{\mathrm{ply}} A_{\mathrm{ply}} B_{\mathrm{ply}}^2 A_{\mathrm{ply}} \overset{e}{\sim} A_{\mathrm{ply}} B_{\mathrm{ply}}^2 A_{\mathrm{ply}} B_{\mathrm{ply}} A_{\mathrm{ply}}^2 B_{\mathrm{ply}}^2 A_{\mathrm{ply}} . \tag{17}$$

By writing $|A| = c$ and $|B| = z$ we actually mean that for any substitution of $\mathbf{a} \in \mathbb{T}^{(6)}$ we have $|A_{\mathrm{ply}}|(\mathbf{a}) = c(\mathbf{a})$ and $|B_{\mathrm{ply}}|(\mathbf{a}) = z(\mathbf{a})$ .

*Proof.* Taking the two products $F_{\mathrm{ply}} = \wp^{\ell}(A_{\mathrm{ply}}, B_{\mathrm{ply}})$ and $G_{\mathrm{ply}} = \wp^{\mathrm{r}}(A_{\mathrm{ply}}, B_{\mathrm{ply}})$ respectively for the left and the right hand side of (17), written respectively as $F_{\mathrm{ply}} = (f_{ij})$ and $G_{\mathrm{ply}} = (g_{ij})$, and considering each $f_{ij}$ and $g_{ij}$, $i, j = 1, 2$, as polynomials in $\mathbb{T}[a, b, c, x, y, z]$ we show that each pair $f_{ij}$ and $g_{ij}$ of polynomials are $e$-equivalent. (In fact $F_{\mathrm{ply}}$ and $G_{\mathrm{ply}}$ are just elements of $M_2(\mathbb{T}[a, b, c, x, y, z]$ whose preimages in $\mathbb{T}[\Lambda_1, \Lambda_2]$ under $\mu_{n,m}$ are monomials, cf. Diagram (8).) Note that by this construction all the coefficients of $f_{ij}$ and $g_{ij}$ are constantly 0, thus all are flat polynomials.

In this view we prove that entry-wise

$$F_{\mathrm{ply}} \overset{e}{\sim} G_{\mathrm{ply}} .$$

To do so, for each pair $f_{ij}$ and $g_{ij}$ we write $f_{ij} = h_{ij} + \alpha_{ij}$ and $g_{ij} = h_{ij} + \beta_{ij}$, where $h_{ij}$, $\alpha_{ij}$, and $\beta_{ij}$ are polynomials in $\mathbb{T}[a, b, c, x, y, z]$, and show that each monomial in $\alpha_{ij}$ and $\beta_{ij}$ is inessential with respect to $h_{ij}$.

Recall that by the hypothesis of the lemma,

$$|A_{\mathrm{ply}}| = c \geq ab \quad \text{ and } \quad |B_{\mathrm{ply}}| = z \geq xy . \tag{18}$$

Using this property, whenever a monomial $f_s$ of $f$ is "greater" that $f_t$, that is $f_s(\mathbf{a}) \geq f_t(\mathbf{a})$ for each $\mathbf{a} \in \mathbb{T}^{(6)}$, then $f_t$ is inessential. We call these "greater" monomials ***dominant monomials*** and mark them in the text using the bold font.

We complete the proof by observing the different entries of the matrices $F_{\mathrm{ply}}$ and $G_{\mathrm{ply}}$ and identifying for each case the monomials which make $\alpha_{ij}$ and $\beta_{ij}$ inessential for $h_{ij}$, respectively. These monomials are labeled in $h_{ij}$ by $\underline{\square}^{(\alpha.\#no)}$ and $\overline{\square}^{(\beta.\#no)}$ according to their numbering in $\alpha_{ij}$ and $\beta_{ij}$ which appear in the diagrams below.

To make the next technical exposition clearer, the monomials in the equations below are ordered lexicographically.

$\underline{i=1, j=1}$**:**

$$\begin{aligned}
h_{11} = {} & 1 \oplus ab \oplus bx \oplus ab^2x \oplus bcx \oplus b^2x^2 \oplus ab^3x^2 \oplus b^2cx^2 \oplus b^3x^3 \oplus b^3c\ x^3 \oplus ay \oplus a^2by \oplus acy \oplus xy \oplus \\
& abxy \oplus a^2b^2xy \oplus cxy \oplus abcxy \oplus c^2x\ y \oplus bx^2y \oplus ab^2x^2y \oplus a^2b^3x^2y \oplus bcx^2y \oplus ab^2cx^2y \oplus bc^2x^2\ y \oplus b^2x^3y \oplus \\
& ab^3x^3y \oplus b^2cx^3y \oplus ab^3cx^3y \oplus b^2c^2x^3y \oplus a^2\ y^2 \oplus a^3by^2 \oplus a^2cy^2 \oplus axy^2 \oplus a^2bxy^2 \oplus a^3b^2xy^2 \oplus acx\ y^2 \oplus \\
& a^2bcxy^2 \oplus ac^2xy^2 \oplus x^2y^2 \oplus abx^2y^2 \oplus a^2b^2x^2y^2 \oplus c\ x^2y^2 \oplus abcx^2y^2 \oplus a^2b^2cx^2y^2 \oplus c^2x^2y^2 \oplus abc^2x^2y^2 \oplus \\
& b\ x^3y^2 \oplus ab^2x^3y^2 \oplus bcx^3y^2 \oplus ab^2cx^3y^2 \oplus ab^2c^2x^3\ y^2 \oplus a^3y^3 \oplus a^3cy^3 \oplus a^2xy^3 \oplus a^3bxy^3 \oplus a^2cxy^3 \oplus \\
& a^3bcx\ y^3 \oplus a^2c^2xy^3 \oplus ax^2y^3 \oplus a^2bx^2y^3 \oplus acx^2y^3 \oplus a^2bcx^2\ y^3 \oplus a^2bc^2x^2y^3 \oplus abz \oplus abcz \oplus bxz \oplus ab^2xz \oplus \\
& bcxz \oplus ab^2cx\ z \oplus bc^2xz \oplus b^2x^2z \oplus ab^3x^2z \oplus b^2cx^2z \oplus b^2c^2x^2z \oplus b^3x^3\ z \oplus b^3cx^3z \oplus ayz \oplus a^2byz \oplus acyz \oplus \\
& a^2bcyz \oplus ac^2yz \oplus abxy\ z \oplus a^2b^2xyz \oplus cxyz \oplus abcxyz \oplus a^2b^2cxyz \oplus c^2xyz \oplus abc^2\ xyz \oplus c^3xyz \oplus bx^2yz \oplus \\
& ab^2x^2yz \oplus a^2b^3x^2yz \oplus bcx^2yz \oplus a\ b^2cx^2yz \oplus bc^2x^2yz \oplus ab^2c^2x^2yz \oplus bc^3x^2yz \oplus b^2x^3y\ z \oplus ab^3x^3yz \oplus \\
& b^2cx^3yz \oplus ab^3cx^3yz \oplus b^2c^2x^3yz \oplus a^2y^2\ z \oplus a^3by^2z \oplus a^2cy^2z \oplus a^2c^2y^2z \oplus axy^2z \oplus a^2bxy^2z \oplus a^3\ b^2xy^2z \oplus \\
& acxy^2z \oplus a^2bcxy^2z \oplus ac^2xy^2z \oplus a^2bc^2xy^2\ z \oplus ac^3xy^2z \oplus abx^2y^2z \oplus a^2b^2x^2y^2z \oplus cx^2y^2z \oplus abcx^2\ y^2z \oplus \\
& a^2b^2cx^2y^2z \oplus c^2x^2y^2z \oplus abc^2x^2y^2z \oplus abc^3x^2\ y^2z \oplus a^3y^3z \oplus a^3cy^3z \oplus a^2xy^3z \oplus a^3bxy^3z \oplus a^2cxy^3\ z \oplus \\
& a^3bcxy^3z \oplus a^2c^2xy^3z \oplus abz^2 \oplus a^2b^2z^2 \oplus abcz^2 \oplus ab^2\ xz^2 \oplus bcxz^2 \oplus ab^2cxz^2 \oplus \mathbf{bc^2xz^2} \oplus b^2x^2z^2 \oplus ab^3x^2\ z^2 \oplus \\
& b^2cx^2z^2 \oplus ab^3cx^2z^2 \oplus b^2c^2x^2z^2 \oplus b^3x^3z^2 \oplus b^3c\ x^3z^2 \oplus a^2byz^2 \oplus acyz^2 \oplus a^2bcyz^2 \oplus \mathbf{ac^2yz^2} \oplus abxy\ z^2 \oplus \\
& a^2b^2xyz^2 \oplus abcxyz^2 \oplus a^2b^2cxyz^2 \oplus abc^2xy\ z^2 \oplus c^3xyz^2 \oplus ab^2x^2yz^2 \oplus a^2b^3x^2yz^2 \oplus bcx^2yz^2 \oplus ab^2\ cx^2yz^2 \oplus \\
& bc^2x^2yz^2 \oplus ab^2c^2x^2yz^2 \oplus bc^3x^2yz^2 \oplus a^2\ y^2z^2 \oplus a^3by^2z^2 \oplus a^2cy^2z^2 \oplus a^3bcy^2z^2 \oplus a^2c^2y^2\ z^2 \oplus a^2bxy^2z^2 \oplus \\
& a^3b^2xy^2z^2 \oplus acxy^2z^2 \oplus a^2bcxy^2\ z^2 \oplus ac^2xy^2z^2 \oplus a^2bc^2xy^2z^2 \oplus ac^3xy^2z^2 \oplus a^3y^3\ z^2 \oplus a^3cy^3z^2 \oplus a^2b^2z^3 \oplus \\
& abcz^3 \oplus abc^2z^3 \oplus ab^2xz^3 \oplus a^2\ b^3xz^3 \oplus ab^2cxz^3 \oplus ab^2c^2xz^3 \oplus bc^3xz^3 \oplus ab^3x^2z^3 \oplus b^2\ cx^2z^3 \oplus ab^3cx^2z^3 \oplus \\
& b^2c^2x^2z^3 \oplus a^2byz^3 \oplus a^3b^2y\ z^3 \oplus a^2bcyz^3 \oplus a^2bc^2yz^3 \oplus ac^3yz^3 \oplus a^2b^2xyz^3 \oplus abcx\ yz^3 \oplus a^2b^2cxyz^3 \oplus \\
& abc^2xyz^3 \oplus c^3xyz^3 \oplus abc^3xy\ z^3 \oplus a^3by^2z^3 \oplus a^2cy^2z^3 \oplus a^3bcy^2z^3 \oplus a^2c^2y^2z^3 \oplus a^2\ b^2z^4 \oplus a^2b^2cz^4 \oplus \\
& a^2b^3xz^4 \oplus ab^2cxz^4 \oplus ab^2c^2xz^4 \oplus b\ c^3xz^4 \oplus a^3b^2yz^4 \oplus a^2bcyz^4 \oplus a^2bc^2yz^4 \oplus ac^3yz^4 \oplus a^2\ b^2cz^5 \oplus abc^3z^5
\end{aligned}$$

$$\alpha_{11} = a^3b^2x^2y^3 \oplus ac^2x^2y^3 \oplus a^3b^2yz^2$$

$$\beta_{11} = a^2b^3x^3y^2 \oplus bc^2x^3y^2 \oplus a^2b^3xz^2$$

By Equation (18) there are two dominate monomials for which:

$$\alpha_{11} = a^3b^2x^2y^3 \oplus ac^2x^2y^3 \oplus a^3b^2yz^2 \leq \mathbf{ac^2yz^2}$$

$$\beta_{11} = a^2b^3x^3y^2 \oplus bc^2x^3y^2 \oplus a^2b^3xz^2 \leq \mathbf{bc^2xz^2}\ .$$

Thus, $\alpha_{11}$ and $\beta_{11}$ are inessential for $f_{11}$ and $g_{11}$ respectively, namely $f_{11} \overset{e}{\sim} g_{11}$.

$\underline{i=1, j=2}$**:**

$$\begin{aligned}
h_{12} = {} & \overline{a}^{(\beta.3.)} \oplus a^2b \oplus abx \oplus a^2b^2x \oplus \overline{cx}^{(\beta.4.)} \oplus abcx \oplus ab^2x^2 \oplus bcx^2 \oplus ab^2cx^2 \oplus bc^2x^2 \oplus b^2cx^3 \oplus b^2c^2x^3 \oplus a^2y \oplus \\
& a^3by \oplus a^2cy \oplus axy \oplus a^2bxy \oplus \underline{\overline{a^3b^2xy}}^{(\beta.5.)}_{(\alpha.2.)} \oplus acxy \oplus a^2bcxy \oplus ac^2xy \oplus abx^2y \oplus a^2b^2x^2y \oplus \overline{cx^2y}^{(\beta.2.)} \oplus abcx^2y \oplus \\
& a^2b^2cx^2y \oplus c^2x^2y \oplus abc^2x^2y \oplus \underline{c^3x^2y}_{(\alpha.3.)} \oplus ab^2x^3y \oplus bcx^3y \oplus ab^2cx^3y \oplus bc^2x^3y \oplus ab^2c^2x^3y \oplus bc^3x^3y \oplus a^3y^2 \oplus \\
& a^4by^2 \oplus a^3cy^2 \oplus a^2xy^2 \oplus a^3bxy^2 \oplus a^2cxy^2 \oplus a^3bcxy^2 \oplus a^2c^2xy^2 \oplus ax^2y^2 \oplus a^2bx^2y^2 \oplus a^3b^2x^2y^2 \oplus acx^2y^2 \oplus \\
& a^2bcx^2y^2 \oplus ac^2x^2y^2 \oplus a^2bc^2x^2y^2 \oplus ac^3x^2y^2 \oplus \underline{\overline{abx^3y^2}}^{(\beta.1.)}_{(\alpha.1.)} \oplus a^2b^2x^3y^2 \oplus a^2b^2cx^3y^2 \oplus \underline{\overline{abc^3x^3y^2}}^{(\beta.1.)}_{(\alpha.1.)} \oplus \\
& a^4y^3 \oplus a^4cy^3 \oplus a^3xy^3 \oplus a^4bxy^3 \oplus a^3cxy^3 \oplus a^3c^2xy^3 \oplus a^2x^2y^3 \oplus a^3bx^2y^3 \oplus a^2cx^2y^3 \oplus a^3bcx^2y^3 \oplus a^2c^2x^2y^3 \oplus \\
& a^2c^3x^2y^3 \oplus \underline{a^2bz}_{(\alpha.4.)} \oplus a^2bcz \oplus abxz \oplus a^2b^2xz \oplus \underline{cxz}_{(\alpha.5.)} \oplus abcxz \oplus abc^2xz \oplus ab^2x^2z \oplus bcx^2z \oplus ab^2cx^2z \oplus \\
& bc^2x^2z \oplus bc^3x^2z \oplus b^2cx^3z \oplus b^2c^2x^3z \oplus a^2yz \oplus a^3byz \oplus a^2cyz \oplus a^3bcyz \oplus a^2c^2yz \oplus a^2bxyz \oplus acxyz \oplus \\
& a^2bcxyz \oplus ac^2xyz \oplus a^2bc^2xyz \oplus ac^3xyz \oplus abx^2yz \oplus a^2b^2x^2yz \oplus cx^2yz \oplus abcx^2yz \oplus a^2b^2cx^2yz \oplus abc^2x^2yz \oplus \\
& abc^3x^2yz \oplus c^4x^2yz \oplus ab^2x^3yz \oplus bcx^3yz \oplus ab^2cx^3yz \oplus bc^2x^3yz \oplus ab^2c^2x^3yz \oplus bc^3x^3yz \oplus a^3y^2z \oplus a^4by^2z \oplus \\
& a^3cy^2z \oplus a^3c^2y^2z \oplus a^2xy^2z \oplus a^3bxy^2z \oplus a^2cxy^2z \oplus a^3bcxy^2z \oplus a^2c^2xy^2z \oplus a^2c^3xy^2z \oplus a^2bx^2y^2z \oplus \\
& a^3b^2x^2y^2z \oplus acx^2y^2z \oplus a^2bcx^2y^2z \oplus ac^2x^2y^2z \oplus a^2bc^2x^2y^2z \oplus ac^3x^2y^2z \oplus ac^4x^2y^2z \oplus a^4y^3z \oplus a^4cy^3z \oplus \\
& a^3xy^3z \oplus a^4bxy^3z \oplus a^3cxy^3z \oplus a^3c^2xy^3z \oplus \overline{a^2bz^2}^{(\beta.6.)} \oplus \underline{acz^2}_{(\alpha.6.)} \oplus a^2b^2xz^2 \oplus abcxz^2 \oplus a^2b^2cxz^2 \oplus \\
& abc^2xz^2 \oplus bcx^2z^2 \oplus ab^2cx^2z^2 \oplus bc^2x^2z^2 \oplus ab^2c^2x^2z^2 \oplus bc^3x^2z^2 \oplus b^2cx^3z^2 \oplus b^2c^2x^3z^2 \oplus a^3byz^2 \oplus a^2cyz^2 \oplus \\
& a^3bcyz^2 \oplus a^2c^2yz^2 \oplus a^2bxyz^2 \oplus acxyz^2 \oplus a^2bcxyz^2 \oplus ac^2xyz^2 \oplus a^2bc^2xyz^2 \oplus ac^3xyz^2 \oplus a^2b^2x^2yz^2 \oplus \\
& abcx^2yz^2 \oplus a^2b^2cx^2yz^2 \oplus c^2x^2yz^2 \oplus abc^2x^2yz^2 \oplus \underline{\overline{c^3x^2yz^2}}^{(\beta.2.)}_{(\alpha.3.)} \oplus abc^3x^2yz^2 \oplus c^4x^2yz^2 \oplus a^3y^2z^2 \oplus a^4by^2z^2 \oplus \\
& a^3cy^2z^2 \oplus a^3c^2y^2z^2 \oplus a^3bxy^2z^2 \oplus a^2cxy^2z^2 \oplus a^3bcxy^2z^2 \oplus a^2c^2xy^2z^2 \oplus a^2c^3xy^2z^2 \oplus a^4y^3z^2 \oplus a^4cy^3z^2 \oplus \\
& \underline{a^2bc^2z^3}_{(\alpha.4.)} \oplus \overline{ac^3z^3}^{(\beta.3.)} \oplus abcxz^3 \oplus a^2b^2cxz^3 \oplus abc^2xz^3 \oplus abc^3xz^3 \oplus \overline{c^4xz^3}^{(\beta.4.)} \oplus ab^2cx^2z^3 \oplus bc^2x^2z^3 \oplus
\end{aligned}$$

$ab^2c^2x^2z^3 \oplus bc^3x^2z^3 \oplus a^3byz^3 \oplus a^2cyz^3 \oplus a^3bcyz^3 \oplus a^2c^2yz^3 \oplus a^2c^3yz^3 \oplus \underline{\overline{a^3b^2xyz^3}}^{(\beta.5.)}_{(\alpha.2.)} \oplus a^2bcxyz^3 \oplus$
$ac^2xyz^3 \oplus a^2bc^2xyz^3 \oplus ac^3xyz^3 \oplus ac^4xyz^3 \oplus a^4by^2z^3 \oplus a^3cy^2z^3 \oplus a^3c^2y^2z^3 \oplus a^2bcz^4 \oplus \overline{a^2bc^2z^4}^{(\beta.6.)} \oplus$
$a^2b^2cxz^4 \oplus abc^2xz^4 \oplus abc^3xz^4 \oplus \underline{c^4xz^4}_{(\alpha.5.)} \oplus a^3bcyz^4 \oplus a^2c^2yz^4 \oplus a^2c^3yz^4 \oplus a^2bc^2z^5 \oplus \underline{ac^4z^5}_{(\alpha.6.)}$

$\alpha_{12} = c^2x^3y^2 \oplus abc^2x^3y^2 \oplus a^3b^2xyz \oplus c^3x^2yz \oplus a^2bcz^2 \oplus \mathbf{c^2xz^2} \oplus a^3b^2z^3 \oplus \mathbf{ac^2z^3}$

$\beta_{12} = abcx^3y^2 \oplus c^3x^3y^2 \oplus c^2x^2yz \oplus a^3b^2z^2 \oplus \mathbf{ac^2z^2} \oplus \mathbf{c^3x\ z^2} \oplus a^3b^2xyz^2 \oplus a^2bcz^3$

Inside $\alpha_{12}$, using Equation (18), we have $c^2x^3y^2 \leq \mathbf{c^2xz^2}$ and $a^3b^2z^3 \leq \mathbf{ac^2z^3}$, so they are inessential. By the following diagram we show that the other monomials of $\alpha_{12}$ are inessential with respect to $h_{12}$.

$$
\begin{array}{ccccccccccccc}
 & \alpha.1. & & \alpha.2. & & \alpha.3. & & \alpha.4. & & \alpha.5. & & \alpha.6. \\
\hline
 & abx^3y^2 & & a^3b^2xy & & c^3x^2y & & a^2bz & & cxz & & acz^2 \\
 & | & & | & & | & & | & & | & & | \\
\alpha_{12} = & abc^2x^3y^2 & \oplus & a^3b^2xyz & \oplus & c^3x^2yz & \oplus & a^2bcz^2 & \oplus & c^2xz^2 & \oplus & ac^2z^3 \\
 & | & & | & & | & & | & & | & & | \\
 & abc^3x^3y^2 & & a^3b^2xyz^3 & & c^3x^2yz^2 & & a^2bc^2z^3 & & c^4xz^4 & & ac^4z^5
\end{array}
$$

The upper and the lower rows specify the monomials of $h_{12}$ that make respectively each term of $\alpha_{12}$ to be inessential. The monomials of $h_{12}$ are correspondingly marked above.

Inside $\beta_{12}$, again using Equation (18), we have $c^3x^3y^2 \leq \mathbf{c^3x\ z^2}$ and $a^3b^2z^2 \leq \mathbf{ac^2z^2}$, so these monomials are inessential. The below diagram shows that all the other monomials of $\beta_{12}$ are inessential with respect to $h_{12}$.

$$
\begin{array}{ccccccccccccc}
 & \beta.1. & & \beta.2. & & \beta.3. & & \beta.4. & & \beta.5. & & \beta.6. \\
\hline
 & abx^3y^2 & & cx^2y & & a & & cx & & a^3b^2xy & & a^2bz^2 \\
 & | & & | & & | & & | & & | & & | \\
\beta_{12} = & abcx^3y^2 & \oplus & c^2x^2yz & \oplus & ac^2z^2 & \oplus & c^3x\ z^2 & \oplus & a^3b^2xyz^2 & \oplus & a^2bcz^3 \\
 & | & & | & & | & & | & & | & & | \\
 & abc^3x^3y^2 & & c^3x^2yz^2 & & ac^3z^3 & & c^4xz^3 & & a^3b^2xyz^3 & & a^2bc^2z^4
\end{array}
$$

Accordingly, $f_{12} \overset{e}{\sim} g_{12}$.

$\underline{i = 2, j = 1}$**:** This case is parallel to the case when $i = 1$ and $j = 2$.

$h_{21} = \underline{b}_{(\alpha.3.)} \oplus ab^2 \oplus b^2x \oplus ab^3x \oplus b^2cx \oplus b^3x^2 \oplus ab^4x^2 \oplus b^3cx^2 \oplus b^4x^3 \oplus b^4cx^3 \oplus aby \oplus a^2b^2y \oplus \underline{cy}_{(\alpha.4.)} \oplus$
$abcy \oplus bxy \oplus ab^2xy \oplus \underline{\overline{a^2b^3xy}}^{(\beta.2.)}_{(\alpha.5.)} \oplus bcxy \oplus ab^2cxy \oplus bc^2xy \oplus b^2x^2y \oplus ab^3x^2y \oplus b^2cx^2y \oplus ab^3cx^2y \oplus b^2c^2x^2y \oplus$
$b^3x^3y \oplus ab^4x^3y \oplus b^3cx^3y \oplus b^3c^2x^3y \oplus a^2by^2 \oplus acy^2 \oplus a^2bcy^2 \oplus ac^2y^2 \oplus abxy^2 \oplus a^2b^2xy^2 \oplus \underline{cxy^2}_{(\alpha.2.)} \oplus$
$abcxy^2 \oplus a^2b^2cxy^2 \oplus c^2xy^2 \oplus abc^2xy^2 \oplus \overline{c^3xy^2}^{(\beta.3.)} \oplus bx^2y^2 \oplus ab^2x^2y^2 \oplus a^2b^3x^2y^2 \oplus bcx^2y^2 \oplus ab^2cx^2y^2 \oplus$
$bc^2x^2y^2 \oplus ab^2c^2x^2y^2 \oplus bc^3x^2y^2 \oplus b^2x^3y^2 \oplus ab^3x^3y^2 \oplus b^2cx^3y^2 \oplus ab^3cx^3y^2 \oplus b^2c^2x^3y^2 \oplus b^2c^3x^3y^2 \oplus a^2cy^3 \oplus$
$a^2c^2y^3 \oplus a^2bxy^3 \oplus acxy^3 \oplus a^2bcxy^3 \oplus ac^2xy^3 \oplus a^2bc^2xy^3 \oplus ac^3xy^3 \oplus \underline{\overline{abx^2y^3}}^{(\beta.1.)}_{(\alpha.1.)} \oplus a^2b^2x^2y^3 \oplus a^2b^2cx^2y^3 \oplus$
$\underline{\overline{abc^3x^2y^3}}^{(\beta.1.)}_{(\alpha.1.)} \oplus \overline{ab^2z}^{(\beta.4.)} \oplus ab^2cz \oplus b^2xz \oplus ab^3xz \oplus b^2cxz \oplus ab^3cxz \oplus b^2c^2xz \oplus b^3x^2z \oplus ab^4x^2z \oplus b^3cx^2z \oplus$
$b^3c^2x^2z \oplus b^4x^3z \oplus b^4cx^3z \oplus abyz \oplus a^2b^2yz \oplus \overline{cyz}^{(\beta.5.)} \oplus abcyz \oplus abc^2yz \oplus ab^2xyz \oplus bcxyz \oplus ab^2cxyz \oplus$
$bc^2xyz \oplus ab^2c^2xyz \oplus bc^3xyz \oplus b^2x^2yz \oplus ab^3x^2yz \oplus b^2cx^2yz \oplus ab^3cx^2yz \oplus b^2c^2x^2yz \oplus b^2c^3x^2yz \oplus b^3x^3yz \oplus$
$ab^4x^3yz \oplus b^3cx^3yz \oplus b^3c^2x^3yz \oplus a^2by^2z \oplus acy^2z \oplus a^2bcy^2z \oplus ac^2y^2z \oplus ac^3y^2z \oplus abxy^2z \oplus a^2b^2xy^2z \oplus cxy^2z \oplus$
$abcxy^2z \oplus a^2b^2cxy^2z \oplus abc^2xy^2z \oplus abc^3xy^2z \oplus c^4xy^2z \oplus ab^2x^2y^2z \oplus a^2b^3x^2y^2z \oplus bcx^2y^2z \oplus ab^2cx^2y^2z \oplus$
$bc^2x^2y^2z \oplus ab^2c^2x^2y^2z \oplus bc^3x^2y^2z \oplus bc^4x^2y^2z \oplus a^2cy^3z \oplus a^2c^2y^3z \oplus a^2bxy^3z \oplus acxy^3z \oplus a^2bcxy^3z \oplus$
$ac^2xy^3z \oplus a^2bc^2xy^3z \oplus ac^3xy^3z \oplus \underline{ab^2z^2}_{(\alpha.6)} \oplus \overline{bcz^2}^{(\beta.6.)} \oplus ab^3xz^2 \oplus b^2cxz^2 \oplus ab^3cxz^2 \oplus b^2c^2xz^2 \oplus$
$b^3x^2z^2 \oplus ab^4x^2z^2 \oplus b^3cx^2z^2 \oplus b^3c^2x^2z^2 \oplus b^4x^3z^2 \oplus b^4cx^3z^2 \oplus a^2b^2yz^2 \oplus abcyz^2 \oplus a^2b^2cyz^2 \oplus abc^2yz^2 \oplus$
$ab^2xyz^2 \oplus bcxyz^2 \oplus ab^2cxyz^2 \oplus bc^2xyz^2 \oplus ab^2c^2xyz^2 \oplus bc^3xyz^2 \oplus ab^3x^2yz^2 \oplus b^2cx^2yz^2 \oplus ab^3cx^2yz^2 \oplus$
$b^2c^2x^2yz^2 \oplus b^2c^3x^2yz^2 \oplus acy^2z^2 \oplus a^2bcy^2z^2 \oplus ac^2y^2z^2 \oplus a^2bc^2y^2z^2 \oplus ac^3y^2z^2 \oplus a^2b^2xy^2z^2 \oplus abcxy^2z^2 \oplus$
$a^2b^2cxy^2z^2 \oplus c^2xy^2z^2 \oplus abc^2xy^2z^2 \oplus \underline{\overline{c^3xy^2z^2}}^{(\beta.3.)}_{(\alpha.2.)} \oplus abc^3xy^2z^2 \oplus c^4xy^2z^2 \oplus a^2cy^3z^2 \oplus a^2c^2y^3z^2 \oplus$

$\overline{ab^2c^2z^3}^{(\beta.4.)} \oplus \underline{bc^3z^3}_{(\alpha.3.)} \oplus ab^3xz^3 \oplus b^2cxz^3 \oplus ab^3cxz^3 \oplus b^2c^2xz^3 \oplus b^2c^3xz^3 \oplus ab^4x^2z^3 \oplus b^3cx^2z^3 \oplus b^3c^2x^2z^3 \oplus abcyz^3 \oplus a^2b^2cyz^3 \oplus abc^2yz^3 \oplus abc^3yz^3 \oplus \underline{c^4yz^3}_{(\alpha.4.)} \oplus \underline{\overline{a^2b^3xyz^3}^{(\beta.2.)}}_{(\alpha.5.)} \oplus ab^2cxyz^3 \oplus bc^2xyz^3 \oplus ab^2c^2xyz^3 \oplus bc^3xyz^3 \oplus bc^4xyz^3 \oplus a^2bcy^2z^3 \oplus ac^2y^2z^3 \oplus a^2bc^2y^2z^3 \oplus ac^3y^2z^3 \oplus ab^2cz^4 \oplus \underline{ab^2c^2z^4}_{(\alpha.6.)} \oplus ab^3cxz^4 \oplus b^2c^2xz^4 \oplus b^2c^3xz^4 \oplus a^2b^2cyz^4 \oplus abc^2yz^4 \oplus abc^3yz^4 \oplus \overline{c^4yz^4}^{(\beta.5.)} \oplus ab^2c^2z^5 \oplus \overline{bc^4z^5}^{(\beta.6.)}$

$\alpha_{21} = abcx^2y^3 \oplus c^3x^2y^3 \oplus c^2xy^2z \oplus a^2b^3z^2 \oplus \mathbf{bc^2z^2} \oplus \mathbf{c^3yz^2} \oplus a^2b^3xyz^2 \oplus ab^2cz^3$

$\beta_{21} = c^2x^2y^3 \oplus abc^2x^2y^3 \oplus a^2b^3xyz \oplus c^3xy^2z \oplus ab^2cz^2 \oplus \mathbf{c^2yz^2} \oplus a^2b^3z^3 \oplus \mathbf{bc^2z^3}$

Inside $\alpha_{21}$, again using Equation (18), we have $c^3x^2y^3 \le \mathbf{c^3y\ z^2}$ and $a^2b^3z^2 \le \mathbf{bc^2z^2}$, so these monomials are inessential. The below diagram shows that all the other monomials of $\alpha_{21}$ are inessential with respect to $h_{21}$.

$$\begin{array}{ccccccccccccc}
 & \alpha.1. & & \alpha.2. & & \alpha.3. & & \alpha.4. & & \alpha.5. & & \alpha.6. \\
\hline
 & abx^2y^3 & & cxy^2 & & b & & cy & & a^2b^3xy & & ab^2z^2 \\
 & | & & | & & | & & | & & | & & | \\
\alpha_{21} = & abcx^2y^3 & \oplus & c^2xy^2z & \oplus & bc^2z^2 & \oplus & c^3y\ z^2 & \oplus & a^2b^3xyz^2 & \oplus & ab^2cz^3 \\
 & | & & | & & | & & | & & | & & | \\
 & abc^3x^2y^3 & & c^3xy^2z^2 & & bc^3z^3 & & c^4yz^3 & & a^2b^3xyz^3 & & ab^2c^2z^4
\end{array}$$

Inside $\beta_{21}$, using Equation (18), we have $c^2x^2y^3 \le \mathbf{c^2yz^2}$ and $a^2b^3z^3 \le \mathbf{bc^2z^3}$, so they are inessential. By the following diagram we show that the other monomials of $\beta_{21}$ are inessential with respect to $h_{21}$.

$$\begin{array}{ccccccccccccc}
 & \beta.1. & & \beta.2. & & \beta.3. & & \beta.4. & & \beta.5. & & \beta.6. \\
\hline
 & abx^2y^3 & & a^2b^3xy & & c^3xy^2 & & ab^2z & & cyz & & bcz^2 \\
 & | & & | & & | & & | & & | & & | \\
\beta_{12} = & abc^2x^2y^3 & \oplus & a^2b^3xyz & \oplus & c^3xy^2z & \oplus & ab^2cz^2 & \oplus & c^2yz^2 & \oplus & bc^2z^3 \\
 & | & & | & & | & & | & & | & & | \\
 & abc^3x^2y^3 & & a^2b^3xyz^3 & & c^3xy^2z^2 & & ab^2c^2z^3 & & c^4yz^4 & & bc^4z^5
\end{array}$$

Accordingly, $f_{21} \overset{e}{\sim} g_{21}$.

$\underline{i = 2, j = 2}$**:** This case is parallel to the case when $i = j = 1$.

$h_{22} = ab \oplus a^2b^2 \oplus ab^2x \oplus a^2b^3x \oplus bcx \oplus ab^2cx \oplus ab^3x^2 \oplus b^2cx^2 \oplus ab^3cx^2 \oplus b^2c^2x^2 \oplus b^3cx^3 \oplus b^3c^2x^3 \oplus a^2by \oplus a^3b^2y \oplus acy \oplus a^2bcy \oplus abxy \oplus a^2b^2xy \oplus abcxy \oplus a^2b^2cxy \oplus c^2xy \oplus abc^2xy \oplus ab^2x^2y \oplus a^2b^3x^2y \oplus bcx^2y \oplus ab^2cx^2y \oplus bc^2x^2y \oplus ab^2c^2x^2y \oplus bc^3x^2y \oplus ab^3x^3y \oplus b^2cx^3y \oplus ab^3cx^3y \oplus b^2c^2x^3y \oplus b^2c^3x^3y \oplus a^3by^2 \oplus a^2cy^2 \oplus a^3bcy^2 \oplus a^2c^2y^2 \oplus a^2bxy^2 \oplus a^3b^2xy^2 \oplus acxy^2 \oplus a^2bcxy^2 \oplus ac^2xy^2 \oplus a^2bc^2xy^2 \oplus ac^3xy^2 \oplus abx^2y^2 \oplus a^2b^2x^2y^2 \oplus abcx^2y^2 \oplus a^2b^2cx^2y^2 \oplus abc^2x^2y^2 \oplus c^3x^2y^2 \oplus abc^3x^2y^2 \oplus c^4x^2y^2 \oplus ab^2x^3y^2 \oplus ab^2cx^3y^2 \oplus ab^2c^2x^3y^2 \oplus bc^3x^3y^2 \oplus bc^4x^3y^2 \oplus a^3cy^3 \oplus a^3c^2y^3 \oplus a^3bxy^3 \oplus a^2cxy^3 \oplus a^3bcxy^3 \oplus a^2c^2xy^3 \oplus a^2c^3xy^3 \oplus a^2bx^2y^3 \oplus a^2bcx^2y^3 \oplus a^2bc^2x^2y^3 \oplus ac^3x^2y^3 \oplus ac^4x^2y^3 \oplus a^2b^2z \oplus a^2b^2cz \oplus ab^2xz \oplus a^2b^3xz \oplus bcxz \oplus ab^2cxz \oplus ab^2c^2xz \oplus ab^3x^2z \oplus b^2cx^2z \oplus ab^3cx^2z \oplus b^2c^2x^2z \oplus b^2c^3x^2z \oplus b^3cx^3z \oplus b^3c^2x^3z \oplus a^2byz \oplus a^3b^2yz \oplus acyz \oplus a^2bcyz \oplus a^2bc^2yz \oplus a^2b^2xyz \oplus abcxyz \oplus a^2b^2cxyz \oplus c^2xyz \oplus abc^2xyz \oplus abc^3xyz \oplus ab^2x^2yz \oplus a^2b^3x^2yz \oplus bcx^2yz \oplus ab^2cx^2yz \oplus bc^2x^2yz \oplus ab^2c^2x^2yz \oplus bc^3x^2yz \oplus bc^4x^2yz \oplus ab^3x^3yz \oplus b^2cx^3yz \oplus ab^3cx^3yz \oplus b^2c^2x^3yz \oplus b^2c^3x^3yz \oplus a^3by^2z \oplus a^2cy^2z \oplus a^3bcy^2z \oplus a^2c^2y^2z \oplus a^2c^3y^2z \oplus a^2bxy^2z \oplus a^3b^2xy^2z \oplus acxy^2z \oplus a^2bcxy^2z \oplus ac^2xy^2z \oplus a^2bc^2xy^2z \oplus ac^3xy^2z \oplus ac^4xy^2z \oplus a^2b^2x^2y^2z \oplus abcx^2y^2z \oplus a^2b^2cx^2y^2z \oplus abc^2x^2y^2z \oplus c^3x^2y^2z \oplus abc^3x^2y^2z \oplus c^4x^2y^2z \oplus c^5x^2y^2z \oplus a^3cy^3z \oplus a^3c^2y^3z \oplus a^3bxy^3z \oplus a^2cxy^3z \oplus a^3bcxy^3z \oplus a^2c^2xy^3z \oplus a^2c^3xy^3z \oplus abcz^2 \oplus a^2b^2cz^2 \oplus abc^2z^2 \oplus ab^2cxz^2 \oplus \mathbf{bc^2xz^2} \oplus ab^2c^2xz^2 \oplus bc^3xz^2 \oplus b^2cx^2z^2 \oplus ab^3cx^2z^2 \oplus b^2c^2x^2z^2 \oplus b^2c^3x^2z^2 \oplus b^3cx^3z^2 \oplus b^3c^2x^3z^2 \oplus a^2bcyz^2 \oplus \mathbf{ac^2yz^2} \oplus a^2bc^2yz^2 \oplus ac^3yz^2 \oplus a^2b^2xyz^2 \oplus abcxyz^2 \oplus a^2b^2cxyz^2 \oplus c^2xyz^2 \oplus abc^2xyz^2 \oplus c^3xyz^2 \oplus abc^3xyz^2 \oplus c^4xyz^2 \oplus a^2b^3x^2yz^2 \oplus ab^2cx^2yz^2 \oplus bc^2x^2yz^2 \oplus ab^2c^2x^2yz^2 \oplus bc^3x^2yz^2 \oplus bc^4x^2yz^2 \oplus a^2cy^2z^2 \oplus a^3bcy^2z^2 \oplus a^2c^2y^2z^2 \oplus a^2c^3y^2z^2 \oplus a^3b^2xy^2z^2 \oplus a^2bcxy^2z^2 \oplus ac^2xy^2z^2 \oplus a^2bc^2xy^2z^2 \oplus ac^3xy^2z^2 \oplus ac^4xy^2z^2 \oplus a^3cy^3z^2 \oplus a^3c^2y^3z^2 \oplus a^2b^2cz^3 \oplus abc^2z^3 \oplus abc^3z^3 \oplus ab^2cxz^3 \oplus bc^2xz^3 \oplus ab^2c^2xz^3 \oplus bc^3xz^3 \oplus bc^4xz^3 \oplus ab^3cx^2z^3 \oplus b^2c^2x^2z^3 \oplus b^2c^3x^2z^3 \oplus a^2bcyz^3 \oplus ac^2yz^3 \oplus a^2bc^2yz^3 \oplus ac^3yz^3 \oplus ac^4yz^3 \oplus a^2b^2cxyz^3 \oplus abc^2xyz^3 \oplus c^3xyz^3 \oplus abc^3xyz^3 \oplus c^4xyz^3 \oplus c^5xyz^3 \oplus a^3bcy^2z^3 \oplus a^2c^2y^2z^3 \oplus a^2c^3y^2z^3 \oplus abc^2z^4 \oplus abc^3z^4 \oplus ab^2c^2xz^4 \oplus bc^3xz^4 \oplus bc^4xz^4 \oplus a^2bc^2yz^4 \oplus ac^3yz^4 \oplus ac^4yz^4 \oplus abc^3z^5 \oplus c^5z^5$

$\alpha_{22} = a^2b^3x^3y^2 \oplus bc^2x^3y^2 \oplus a^2b^3xz^2$

$\beta_{22} = a^3b^2x^2y^3 \oplus ac^2x^2y^3 \oplus a^3b^2yz^2$

By Equation (18) there are two dominant monomials for which:

$\alpha_{22} = a^2b^3x^3y^2 \oplus bc^2x^3y^2 \oplus a^2b^3xz^2 \leq \mathbf{bc^2xz^2}$

$\beta_{22} = a^3b^2x^2y^3 \oplus ac^2x^2y^3 \oplus a^3b^2yz^2 \leq \mathbf{ac^2yz^2}$

Thus, $\alpha_{22}$ and $\beta_{22}$ are inessential for $f_{22}$ and $g_{22}$ respectively, namely $f_{22} \overset{e}{\sim} g_{22}$.

Composing all together, $A_{\mathrm{ply}} \overset{e}{\sim} B_{\mathrm{ply}}$ to complete the proof of Lemma 4.6. $\square$

**Note.** *To perform the symbolic computations in the proof of Lemma 4.6 we were assisted by the mathematical software Wolfram Mathematica* 6.

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